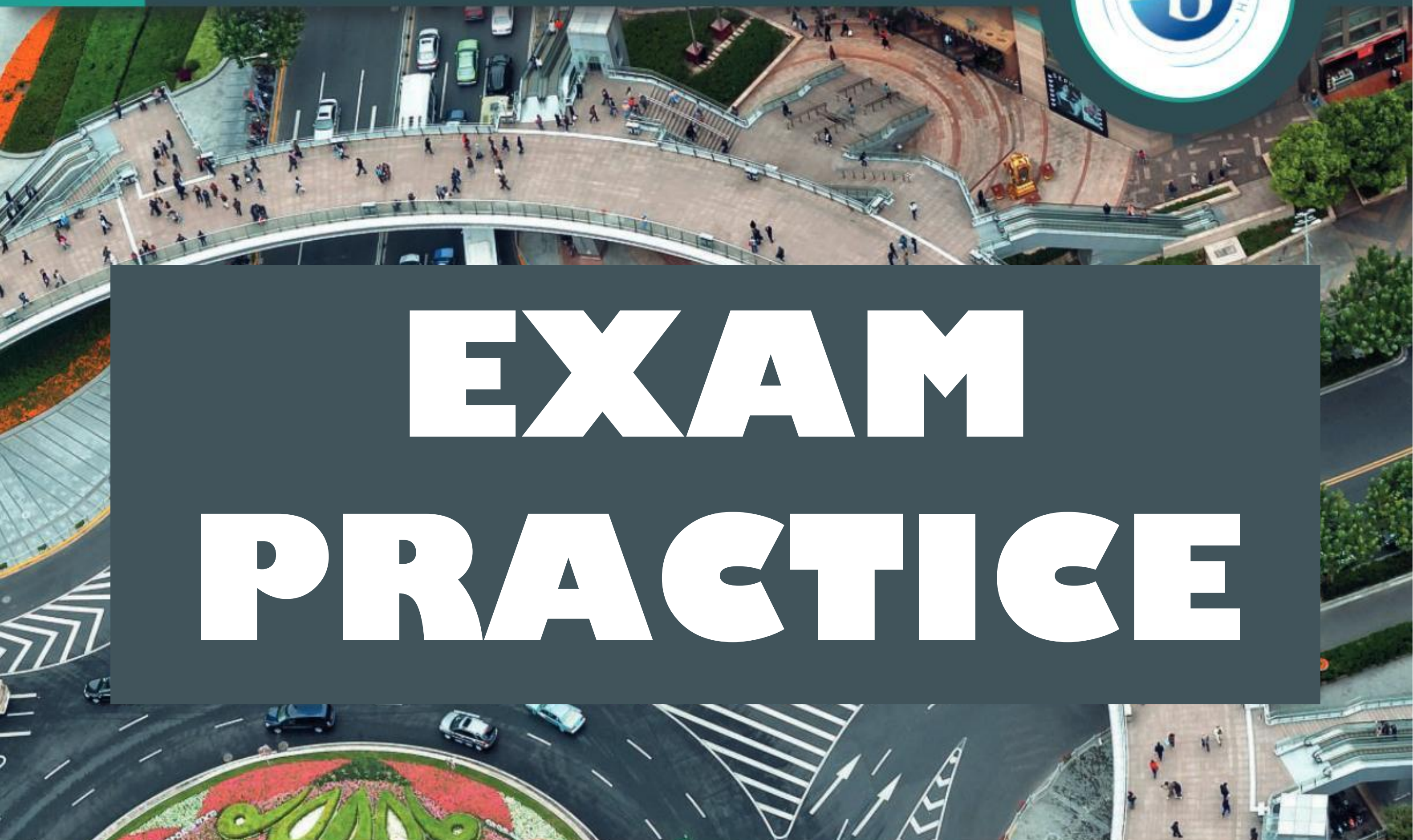
OXFORD IB DIPLOMA PROGRAMME



EXAM PRACTICE

MATHEMATICS:
APPLICATIONS AND INTERPRETATION

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

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Exam practice: chapters 1-3

1 P1: A snooker ball is hit a distance of $70\,cm$ and bounces off the cushion on the side of the table. The angle between the path of the ball and the cushion before it rebounds is 40°, which is the same angle between the cushion and the new path of the ball after it has rebounded.

After the ball rebounds, it travels another $60\,cm$ before it stops. Find the straight-line distance between the point where the ball is hit and the point where it stops. (4)

2 P1: Josie is lying on the top of a vertical sea cliff that is $50m$ high. She can see her mother in kayak on the sea. The angle of depression from Josie to her mother is 40°.

a Find the distance between the kayak and the base of the cliff. (3)

Josie's father is in a hang glider $100m$ directly above Josie.

b Find the angle of depression from the hang glider to the kayak. (3)

3 P1: On open range land, natural resources are owned by the nearest ranch. In a coordinate system with lengths measured in kilometres, there are ranches A, B, C and D situated at points $(19,13), (14,24), (25,25)$ and $(27,20)$ respectively. There is a well at point $(20,20)$ that all 4 ranchers claim to own. Determine (with a reason) which ranch the well really belongs to. (5)

4 P2: A data set has a mean of $\bar{x}$, mode of m, median of Q_2, lower quartile of Q_1, upper quartile of Q_3 and variance of s_n^2.

All the values in the original data set have 2 added to them.

a Find each of the following for the new set of data. Express your answers in terms of the parameters $\bar{x}$, m, Q_2, Q_1, Q_3 and s_n^2 from the original data set.

i the mean **ii** the mode **iii** the median

iv the interquartile range **v** the variance. (5)

All the values from the original data set are now multiplied by −3.

b Find each of the following for the new set of data. Express your answers in terms of the parameters $\bar{x}$, m, Q_2, Q_1, Q_3 and s_n^2 from the original data set.

i the mean **ii** the mode **iii** the median

iv the lower quartile **v** upper quartile **vi** the variance. (6)

c i All the values from the original data set have 4 added to them, and are then multiplied by 5. Find an expression for the new variance.

ii All the values in the original data set are multiplied by 5, and then have 4 added to them. Find an expression for the new variance.

iii Using your answers to parts **i** and **ii**, comment on how variance is affected by both addition and multiplication of each data piece. (3)

5 P1: The planet Saturn travels in a circular orbit of radius $1.434 \times 10^9 km$ around the sun.

a Find the distance Saturn travels in one revolution of the sun, giving your answer in standard form correct to two significant figures. (3)

b Find the area contained within one revolution of Saturn's orbit, giving your answer in standard form correct to three significant figures. (3)

It takes 29 Earth years for Saturn to complete one revolution.

c Find the average speed of Saturn in kilometres per second, giving your answer correct to the nearest integer. You may assume that one Earth year is 365 days. (3)

6 P1: A pendulum of length $20 cm$ swings through an angle of 10°.

a Find the distance that the bottom of the pendulum travels in one complete swing. (3)

b Find the area of the sector which the pendulum sweeps out. (3)

7 P1: A promenade runs in a straight line from West to East and is $10km$ long. The distance om the west end is denoted by x and is measured in kilometres.

There are five ice-cream kiosks situated at $x = 0, 2, 3, 6$ and 10. Each kiosk owns the distances along the promenade that are nearer to that kiosk than to any other.

Rachel is going to set up a new ice-cream kiosk and place it along the promenade.

a Initially, she decides to put her kiosk as far as possible from any other kiosk. State the x value where she will place her kiosk. (1)

b Find the set of other x values that would mean Rachel owned the same amount of promenade distance as she would with your answer in **a**. (2)

c Explain why the value of x found in **a** would be the best if Rachel suspected that yet another ice-cream kiosk might be built on the promenade. (2)

Instead of buying a new kiosk, Rachel now decides to buy one of the existing five kiosks.

d State (with a reason) which one she should choose. (2)

8 P2: A cylindrical hole of depth $10 cm$ and radius $4 cm$ is drilled into a piece of metal. The hole filled with water. A right circular cone of height $10 cm$ and radius $4 cm$ is turned upside nd placed in the hole with its vertex at the bottom of the hole and its base level with the top of the hole. This displaces some of the water.

a Find the volume of water remaining in the hole. (4)

b Find the surface area of the space that is occupied by the remaining water. (It is given that the surface area of a cone is πrl where l is the slant height) (7)

9 P2: The perpendicular bisector between the points $A = (2, 3)$ and $B = (x, y)$ has the equation $= 2x + 1$. Find the exact coordinates of the point B. (10)

10 P2: Two circles, both of radius $5 cm$, overlap one another. The distance between the two intersecting points is $8 cm$. Find the value of the area that is common to both circles. (10)

11 P1: Let l_1 and l_2 be two straight lines with equations $3x + 4y = 1$ and $7x - 5y = -2$ respectively.

a Determine whether the lines are parallel. (3)

b Given another straight line l_3 with equation $5kx - 7y = -4$ where $k \in \mathbb{R}$, find $k \in \mathbb{R}$ such that l_2 and l_3 are perpendicular. (3)

12 P2: A solid metallic cylinder of height $h = 10cm$ and base radius of $r = 3cm$ is melted down to form some spheres, each of which has radius $7mm$. Work out the maximum number of spheres that can be made from the cylinder. (7)

13 P1: Calculate the length of the chord AB of a circle with radius $AO = 10cm$, where O is the centre, given that the arc AB subtends an angle of 120° at the centre.

Give your answer in the form $k\sqrt{3}$, where k is an integer to be found. (4)

14 P1: Let $A = (4, -4, 5)$ and $B = (-2, 11, 0)$ be two points, and $\mathbf{u} = \begin{pmatrix} 7 \\ -8 \\ 3 \end{pmatrix}$ be a vector.

Let l_1 be the straight line that passes though points A and B, and l_2 be the straight line that passes though point A and is parallel to vector $\mathbf{u}$.

a Find vector equations of the lines l_1 and l_2. (5)

b The point C has coordinates $C = (-3, 4, 2)$.

i Verify that the point C lies on l_2. (1)

ii Find the distance CB. Give your answer in simplified surd form. (2)

iii Calculate the angle $C\hat{B}A$ to the nearest degree. (4)

15 P1: **a** Prove that the following straight lines $\begin{aligned} l_1 &: x + 4y = 5 \\ l_2 &: 3x - 2y = 1 \\ l_3 &: 7x - 8y + 1 = 0 \end{aligned}$ all intersect at a single point, and find the coordinates of this point. (5)

b Find the value of the parameter k such that the point of intersection lies on the line $kx - 5ky = -9$. (2)

16 P2: The following eight straight lines can be arranged as four pairs of perpendicular lines. Identify the four pairings, and give reasons for your answers. (10)

$y = 3x + 4$ $\quad$ $x = 3$ $\quad$ $2x - y + 3 = 0$

$10x + 2y + 1 = 0$ $\quad$ $y = 7$ $\quad$ $y = \frac{-1}{3}x + 7$

$y = \frac{-1}{2}x - 2$ $\quad$ $x - 5y + 4 = 0$

17 P1: Consider a solid hemisphere with radius R. The hemisphere has volume $V = 32\pi cm^3$. Calculate

a the radius of the hemisphere (2)

b the surface area S of the hemisphere, giving your answer in terms of π. (2)

18 P2: Let l_1 and l_2 be two straight lines with vector equations:

$\mathbf{r}_1 = \begin{pmatrix} 8 \\ 6 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}$ and $\mathbf{r}_2 = \begin{pmatrix} -4 \\ 0 \\ 11 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$ respectively, where $\lambda, \mu \in \mathbb{R}$ are parameters.

a Prove that the lines above intersect at a point P, and find the coordinates of P. (4)

b Calculate the angle θ between these two lines, giving your answer to the nearest degree. (2)

19 P2: Points A and B have coordinates: $A = (5, 4, -7)$ and $B = (3, -2, 6)$.

a Find a vector equation for the line passing through A and B. (2)

b Point C lies on the line on the line through A and B, and is such that $\overrightarrow{OC}$ is perpendicular to the line. Show that the coordinates of C are $C = \left(\frac{795}{209}, \frac{86}{209}, \frac{162}{209}\right)$.

(3)

c Work out the area of the triangle OAC (where O is the origin) correct to the nearest square unit. (4)

d Work out the area of the triangle OAB correct to the nearest square unit. (3)

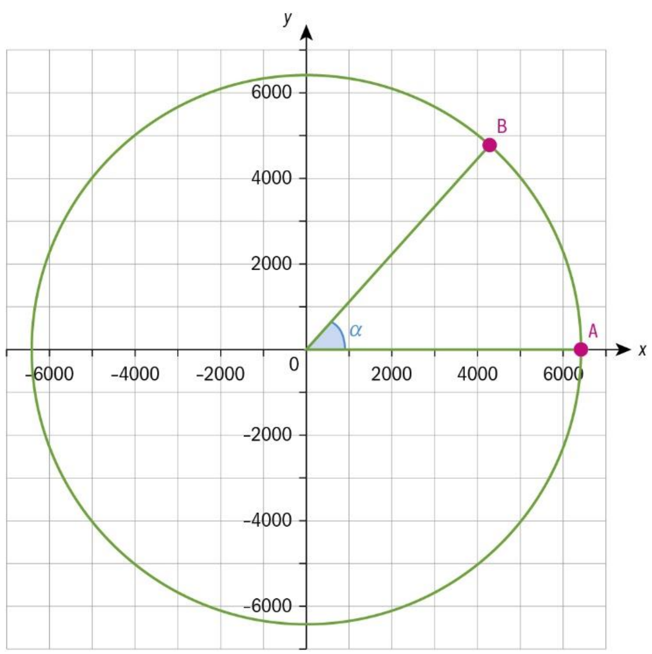
20 P3: The aim of the question is to find the shortest distance between Lima in Peru and Tokyo in Japan, to an appropriate degree of accuracy.

A point on the surface of the earth can be described using two angles. Its longitude measures how far east the point is from the prime meridian, 0°. Its latitude measures how far north the point is from the equator. A negative longitude indicates the point is west of the prime meridian and a negative latitude indicates it is south of the equator.

Let the earth be modelled as a sphere of radius 6370 km. Let the centre of the earth be the origin (O) of a coordinate system in which the z axis passes through the two poles, and the x and y axes lie in the plane containing the equator with the x axis passing through the point on the equator with longitude $0°$.

a The diagram shows the equator set in the given coordinate system. You are looking at the x and y axes from above.

Two towns A and B lie on the equator. A has coordinates $(6370, 0)$. The longitude of town B is given by the angle α. α is positive for counter clockwise rotations and negative for clockwise rotations.



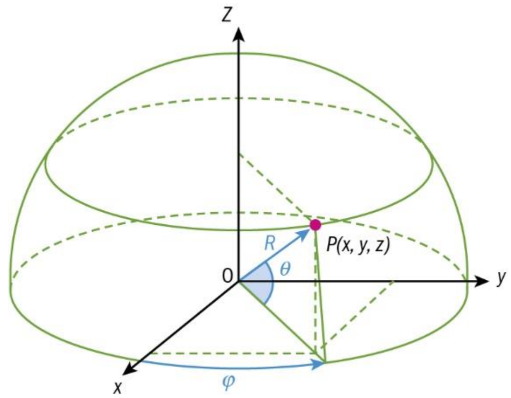
i When $\alpha = 50$� show that the coordinates of B, to three significant figures, are $(4090, 4880)$.

ii Find the shortest distance around the surface of the earth from A to B.

iii Find the shortest straight-line distance between A and B.

iv Find the percentage error in using the straight-line distance as an approximation for the distance along the surface of the earth. (9)

Consider the point P, which lies on the surface of the sphere at the point with latitude θ and longitude ϕ, as shown in the diagram below. Let the radius of the earth be R.



b **i** Write down the z coordinate of P in terms of R and θ.

ii Find the coordinates of P in terms of R, θ and ϕ. (4)

In this model, Lima in Peru lies at the point with latitude −12.05 and longitude −77.04.

c Show that the coordinates of Lima are $(1397, -6071, -1329)$. (2)

In this same model, Tokyo lies at the point with latitude 35.69 and longitude 139.70. This point has coordinates $(-3945, 3346, 3717)$.

Let Lima be at the point L and Tokyo be at the point T.

d **i** Write down $|\overrightarrow{OL}|$

ii Find the angle $L\hat{O}T$

iii Hence find the shortest distance around the surface of the earth between Lima and Tokyo. (8)

In reality, the earth is not a perfect sphere and the distance from the centre of the earth to the surface is between 6353 km and 6384 km.

e **i** Find upper and lower bounds for the distance between Lima and Tokyo along the shortest route, assuming that both are the same distance from the centre of the earth.

ii Hence give the distance between the two cities to an appropriate degree of accuracy. (4)

Answers

1 Let the ball start at A , rebound at B and stop at C . Then $\hat{B} = 180 - 2 \times 40 = 100$ A1

$c^2 = 70^2 + 60^2 - 2 \times 70 \times 60 \cos 100 \Rightarrow c = 99.8\,cm(3sf)$ M1A1A1

2 **a** $90 - 40 = 50$ $\quad \frac{x}{50} = \tan 50 \Rightarrow x = 59.587...$ A1M1

Distance is $59.6m(3sf)$ A1

b $\tan\alpha = \frac{59.587...}{150} \Rightarrow \alpha = 21.66...$ M1A1

angle of depression $= 90 - 21.66 = 68.3^\circ (3sf)$ A1

3 Distances are $\sqrt{1^2 + 7^2} = \sqrt{50}, \sqrt{6^2 + 4^2} = \sqrt{52}, \sqrt{5^2 + 5^2} = \sqrt{50}, \sqrt{7^2 + 0} = 7$ respectively

A1A1A1A1

So the well belongs to ranch D. R1

4 **a** **i** $\bar{x} + 2$ **ii** $m + 2$ **iii** $Q_2 + 2$

iv $Q_3 - Q_1$ **v** s_n^2. A1A1A1A1A1

b **i** $-3\bar{x}$ **ii** $-3m$ **iii** $-3Q_2$

iv $-3Q_3$ **v** $-3Q_1$ **vi** $9s_n^2$. A1A1A1A1A1A1

c **i** $25s_n^2$ **ii** $25s_n^2$ A1A1

iii Variance affected only when each data piece is multiplied by a constant, not by adding, so answers to **i** and **ii** are the same. R1

5 **a** $2 \times \pi \times 1.434 \times 10^9 = 9.0 \times 10^9\,km$ M1A1A1

b $\pi\left(1.434 \times 10^9\right)^2 = 6.46 \times 10^{18}\,km^2$ M1A1A1

c $\frac{9.01 \times 10^9}{29 \times 365 \times 24 \times 60 \times 60} \simeq 10\,km\,s^{-1}$ M1A1A1

6 **a** $2\pi \times 20 \times \frac{10}{360} = 3.49\,cm(3sf)$ M1A2

b $\pi \times 20^2 \times \frac{10}{360} = 34.9\,cm^2(3sf)$ M1A2

7 **a** $x = 8$ A1

b She would control 2 km. This is also achieved by having $6 < x < 10$ R1A1

c If a new kiosk was placed halfway between Rachel and one of the original kiosks, Rachel would lose half the distance on that side. Thus, it is best to have the two distances that she owns (one on either side) equal, giving $x = 8$. R1R1

d Best to take over the one at $x = 6$ as it controls $3.5km$ which is more than any of the others. A1R1

8 **a** Volume of cylinder minus volume of cone $= \pi \times 4^2 \times 10 - \frac{\pi \times 4^2 \times 10}{3} = 335cm^3 (3sf)$ R1M1A2

b Area will consist of a circle at the base of the cylinder, the curved surface of the cylinder, and the curved surface of the cone. (R1)

$l = \sqrt{10^2 + 4^2} = \sqrt{116}$ M1A1

Area $= \pi \times 4^2 + 2\pi \times 4 \times 10 + \pi \times 4 \times \sqrt{116} = 437 cm^2\ (3sf)$ M1A1A2

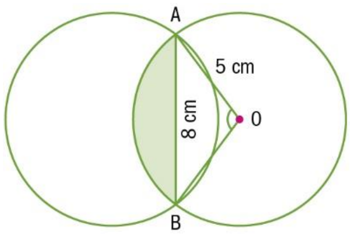
9 Line through A and B has equation $y = -\frac{1}{2}x + c$. R1

Passes through $A \Rightarrow 3 = -\frac{1}{2} \times 2 + c \Rightarrow c = 4$, line is $y = -\frac{1}{2}x + 4$. M1A1

Intersection of $y = 2x + 1$ and $y = -\frac{1}{2}x + 4$ is $\left(\frac{6}{5}, \frac{17}{5}\right)$ M1A1A1

$\frac{2+x}{2} = \frac{6}{5} \Rightarrow x = \frac{2}{5} \qquad \frac{3+y}{2} = \frac{17}{5} \Rightarrow y = \frac{19}{5}$ M1A1M1A1

10



A sketch (or reasoning) shows that, by symmetry of the two circles,

overlapping area = 2 x [(area of sector AOB) − (area of triangle AOB)]. R1

Let the angle AOB (as shown in the sketch) be 2α M1

$\sin\alpha = \frac{4}{5} \Rightarrow \alpha = 53.13^\circ\ldots$ M1A1

overlapping area $= 2\left(\pi \times 5^2 \times \frac{2 \times 53.13^\circ\ldots}{360} - \frac{1}{2} \times 5 \times 5 \times \sin\left(2 \times 53.13^\circ\ldots\right)\right)$ M1A1M1A1

$= 22.4 cm^2 (3sf)$ A2

11 a Rearrange to make y the subject: $l_1 : y = \frac{1}{4} - \frac{3x}{4}$, $l_2 : y = \frac{2}{5} + \frac{7x}{5}$ M1

gradients are $\frac{-3}{4}$ and $\frac{7}{5}$, which are not equal R1

Hence, not parallel. A1

b Rearrange to make y the subject: $l_3 : y = \frac{4}{7} + \frac{5kx}{7}$ (M1)

l_2 and l_3 are perpendicular $\Rightarrow \frac{5k}{7} \times \frac{7}{5} = -1$ A1

$k = -1$ A1

12 $V_{cylinder} = \pi \times 3^2 \times 10 = 90\pi cm^3$ M1A1

$V_{sphere} = \frac{4}{3} \times \pi \times (0.7)^3 = \frac{343\pi}{75 \times 10} cm^3$ M1A1

Answer $= \frac{90\pi}{\left(\frac{343\pi}{75 \times 10}\right)} = 196.793$ M1A1

197 spheres to the nearest sphere. A1

13 $A\hat{O}C = 120^\circ$ A1

The cosine rule: $AB = \sqrt{10^2 + 10^2 - 2 \times 10 \times 10 \times \cos(120^\circ)}$ M1A1

$AB = 10\sqrt{3} cm$. A1

14 a Attempt to find the parallel vector of l_1 : $\mathbf{u} = \begin{pmatrix} 6 \\ -15 \\ 5 \end{pmatrix}$ M1

Use either $\overrightarrow{OA} = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix}$ or $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 11 \\ 0 \end{pmatrix}$ M1

$\mathbf{r}_1 = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -15 \\ 5 \end{pmatrix}$ or equivalent A1

$\overrightarrow{OA} = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix}$ M1

$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -8 \\ 3 \end{pmatrix}$, or equivalent. A1

b i For $\mu = -1$ from line l_2 we obtain $\overrightarrow{OC} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ R1

ii $\left|\overrightarrow{CB}\right| = \sqrt{(-3-(-2))^2 + (4-11)^2 + (2-0)^2}$ M1

$\left|\overrightarrow{CB}\right| = 3\sqrt{6}$ A1

iii $\overrightarrow{CB} = \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}$ M1

$\overrightarrow{BA} = \begin{pmatrix} 6 \\ -15 \\ 5 \end{pmatrix}$ M1

Write the dot product: $\begin{pmatrix}1\\7\\-2\end{pmatrix}\cdot\begin{pmatrix}6\\-15\\5\end{pmatrix}=\sqrt{1^2+7^2+(-2)^2}\times\sqrt{6^2+(-15)^2+5^2}\times\cos(C\hat{B}A)$

M1

$C\hat{B}A=151^\circ$, to the nearest degree. A1

15 a Attempt to find the common point of two lines. M1

e.g. Common point of l_1, l_2 is $(1,1)$ A1

Substitution of above point into the third line. M1

e.g. $7\times1-8\times1+1=0$ A1

The point satisfies the other line, so they all intersect at the point with coordinates (1, 1). R1

b Substitution of $(1,1)$ in the fourth line. M1

Solve for k $\Rightarrow k=\frac{9}{4}$ A1

16 $y=3x+4$ and $y=\frac{-1}{3}x+7$; gradients of 3 and $\frac{-1}{3}$ A1R1

$2x-y+3=0\Rightarrow y=2x+3$ and $y=\frac{-1}{2}x-2$; gradients of 2 and $\frac{-1}{2}$ M1A1R1

$10x+2y+1=0\Rightarrow y=-5x-\frac{1}{2}$ and $x-5y+4=0\Rightarrow y=\frac{1}{5}x+\frac{4}{5}$; gradients of -5 and $\frac{1}{5}$ M1A1R1

$x=3$ and $y=7$; vertical and horizontal A1R1

17 a $V=\frac{2}{3}\pi R^3$ M1

Obtain $R=48^{1/3}cm$ A1

b $S=\frac{4\pi R^2}{2}+\pi R^2$ M1

(Since it is a non − hollow hemisphere, we need the half of the spherical surface area plus the area or the circle with radius R.)

$S=12\times6^{2/3}\pi cm^2$ A1

18 a $\begin{pmatrix}8+3\lambda\\6-3\lambda\\-9-\lambda\end{pmatrix}=\begin{pmatrix}-4+2\mu\\4\mu\\11-6\mu\end{pmatrix}$ at point P. M1

Obtain from the first two equations: $\begin{matrix}\lambda=-2\\\mu=3\end{matrix}$ A1

Check the validity of the third equation by substituting $\begin{matrix}\lambda=-2\\\mu=3\end{matrix}$ in:

$-9-(-2)=11-6(3)=-7$ M1

$P = (2, 12, -7)$ A1

b $\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} = 0$ M1

so the lines are orthogonal and $\theta = 90^\circ$, to the nearest degree. A1

19 a Direction of line through A and B is $\begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -13 \end{pmatrix}$ A1

$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 6 \\ -13 \end{pmatrix}, \ \lambda \in \mathbb{R}$ A1

b Because C lies on the line through A and B, $\overrightarrow{OC} = \begin{pmatrix} 5 + 2\lambda \\ 4 + 6\lambda \\ -7 - 13\lambda \end{pmatrix}$ M1

$\begin{pmatrix} 5 + 2\lambda \\ 4 + 6\lambda \\ -7 - 13\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ -13 \end{pmatrix} = 0$ M1

Obtain: $\lambda = \dfrac{-125}{209}$ A1

$C = \left(\dfrac{795}{209}, \dfrac{86}{209}, \dfrac{162}{209}\right)$ AG

c $\left|\overrightarrow{OC}\right| = \sqrt{\dfrac{3185}{209}}$ M1

$\overrightarrow{AC} = \begin{pmatrix} \dfrac{-250}{209} \\ -\dfrac{750}{209} \\ \dfrac{1625}{209} \end{pmatrix}$ M1

$\left|\overrightarrow{AC}\right| = \sqrt{\dfrac{15625}{209}}$ M1

Area $= \dfrac{1}{2} \times \sqrt{\dfrac{3185}{209}} \times \sqrt{\dfrac{15625}{209}} \approx 17$ square units, correct to the nearest square unit. A1

d $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -6 \\ 13 \end{pmatrix}$ M1

$\left|\overrightarrow{AB}\right| = \sqrt{209}$ M1

Area $= \dfrac{1}{2} \times \sqrt{\dfrac{3185}{209}} \times \sqrt{209} \approx 28$ square units, correct to the nearest square unit. A1

20 Answers are generally given to 4 significant figures but accept any answer which rounds correctly to three significant figures.

a **i** Coordinates of B are $(6370\cos 50\text{�}, 6370\sin 50\text{�})$ M1A1

$\approx (4090, 4880)$

ii $\dfrac{50}{360} \times 2\pi \times 6370 \approx 5560$ M1A1

iii $\sqrt{(6370-4090)^2 + 4880^2}$ M1A1

≈ 5380 km A1

iv Percentage error = $\dfrac{5560-5380}{5560} \times 100 = 3.2\%$ M1A1

b **i** $R\sin\theta$ A1

ii Projection of P onto the plane of the equator is a

distance $R\cos\theta$ from O (M1)

Coordinates are $(R\cos\theta\cos\phi, R\cos\theta\sin\phi, R\sin\theta)$ A1A1

c **i** $(6370\cos(-12.05)\cos(-77.04), 6370\cos(-12.05)\sin(-77.04), 6370\sin(-12.05))$

M1A1

$\approx (1397, -6071, -1330)$ AG

d **i** 6370 km A1

ii $\begin{pmatrix} 1397 \\ -6071 \\ -1330 \end{pmatrix} \cdot \begin{pmatrix} -3945 \\ 3346 \\ 3717 \end{pmatrix} = -3.07681 \times 10^7$ M1(A1)

Using the fact that both cities lie on the surface of the earth, and hence have magnitude 6370, gives

$\cos(\mathrm{L\hat{O}P}) \approx -0.758$ M1A1

$\approx 139.3^{\circ}$ A1

Note: It is also possible to solve **ii** by finding the straight-line distance between Lima and Tokyo, and use the cosine rule or right-angled trigonometry.

iii $\dfrac{139.3}{360} \times 2\pi \times 6370$ (M1)

$\approx 15\,490$ km A1

e **i** Distances form arcs of a circle subtending the same angle and are hence proportional to the radius. (M1)

Upper bound $15\,487 \times \dfrac{6384}{6370} = 15\,520$ km A1

Lower bound $15\,487 \times \dfrac{6353}{6370} = 15\,450$ km A1

ii Both are approximately 15 500 km to 3 significant figures. A1

Exam practice: chapters 1-7

1 P2: The following table shows the masses of a sample of rabbits from New Zealand.

Mass, w kg	Frequency
$1.5 \le w < 1.6$	9
$1.6 \le w < 1.7$	24
$1.7 \le w < 1.8$	32
$1.8 \le w < 1.9$	62
$1.9 \le w < 2.0$	45
$2.0 \le w < 2.1$	29

a State the modal class (1)

b Find an estimate for the mean mass of the rabbits. (3)

c Find an estimate for the standard deviation. (2)

d Determine the class in which the median mass lies. (3)

Continuous data may be said to be 'normally distributed' if the mean and median are equal, and the range is approximately equal to six standard deviations.

e Determine whether the masses of these rabbits could be normally distributed. (4)

2 P2: The probability of a healthy person catching the flu one winter is 0.02 if they have been vaccinated, and 0.12 if they have not been vaccinated. It is given that 30% of the population have been vaccinated.

a Illustrate the above information on a probability tree diagram. (3)

b Find the probability that a randomly chosen person has been vaccinated and catches the flu. (2)

c Find the probability that a person has not been vaccinated, given that they catch the flu. (3)

3 P1: Two events A and B are independent. It is given that $P(A) = 0.3$ and $P(B) = 0.8$.

a State, with a reason, whether events A and B are mutually exclusive. (2)

b Find the probability of the event:

i $A \cap B$

ii $A \cup B$

iii $A \mid B'$

iv $A' \cap B$ (8)

4 P2: A ball is thrown vertically into the air. The vertical height of the ball above the ground, in metres, may be modelled by the function $f(t) = 2.1 + 14.5t - 4.9t^2$, where t is the time in seconds since the ball was thrown $(t \geq 0)$.

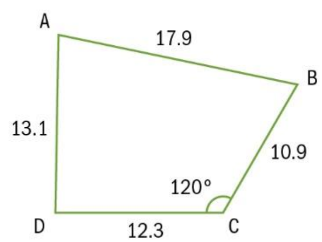
a State the height from which the ball is initially thrown. (1)

b Determine how long the ball is in the air. (2)

c Find the length of time for which the ball is at least 10 m above the point of projection. (4)

5 P1: The angle of elevation to an aeroplane from a control tower 85 m high is 6.5°. Given that the plane is flying at an altitude of 1850 m, find the horizontal distance (in km) of the plane from the control tower correct to three significant figures. (5)

6 P1: In the quadrilateral $ABCD$, $AB = 17.9$ cm, $BC = 10.9$ cm, $CD = 12.3$ cm, and $DA = 13.1$ cm. It is also given that $\angle BCD = 120^\circ$.



a Calculate the size of $\angle DAB$. (6)

b Hence or otherwise, find the area of the quadrilateral $ABCD$. (4)

7 P1: The first three terms in an arithmetic series are given by $5k - 3,\ 19,\ 3k + 1$

a Find the value of k. (3)

b Find the common difference d. (2)

c Find the number of terms required for the sum of the series to equal -1980. (5)

8 P2: Three points A, B and C are indicated by their respective coordinates $A(1,8)$, $B(1,2)$ and $C(10,6)$.

a State the equation of the perpendicular bisector of AB. (1)

b Find the equation of the perpendicular bisector of BC. Give your answer in the form $ax + by + c = 0$. (5)

c Hence find coordinates of the point M which is equidistant from A, B and C. (3)

d Find the equation of the perpendicular bisector of AC. Give your answer in the form $ax + by + c = 0$. (4)

e Draw a Voronoi Diagram for the three points A, B and C. (3)

9 P2: The gravitational field strength on a planet varies inversely with the square of the distance from the centre of the body.

Jupiter has an approximate radius of 7.1492×10^7 m and a gravitational field strength of 24.79 m s^{-2} on its surface.

a Find the gravitational field strength 100 km above Jupiter's surface. (6)

b Find the distance above Jupiter's surface where the gravitational field strength is one-hundredth of that on the surface. Give your answer in standard form correct to three significant figures, in terms of kilometres. (4)

10 P1: a, b, c are consecutive members of an arithmetic sequence with first term a and common difference d.

a Express a, b, c in terms of a_1 and d. (2)

b Prove that $b = \dfrac{a+c}{2}$. (4)

11 P1: Find the sum of

a the first 200 odd positive numbers (2)

b all the odd numbers between 16 and 380 (5)

12 P1: **a** Find the sum of all the integers between 1 and 200. (3)

b Find each of the sums $S_2 = 4 + 8 + \ldots + 200$, $S_3 = 9 + 18 + \ldots + 198$ and $S_4 = 36 + 72 + \ldots + 180$. (9)

c Use parts **a** and **b** to derive the sum of all integers between 1 and 200, except for the multiples of 4 or 9. Justify the method that you use. (3)

13 P2: **a** John invested £5100 for 3 years in an account which paid annual compound interest of 2% per year. Work out the amount of money that John will get after the period of 3 years, giving your answer to the nearest pound. (2)

b Find the number of years (correct to the nearest year) that John must invest £10 000 at an annual rate of 2.5% compound interest if it is to be worth £100 000. (2)

14 P2: The following table illustrates the weekly book sales from an independent book seller on a selection of six different weeks over the course of one year (52 weeks).

Week no.	1	6	10	19	30	50
No. of books sold	400	266	180	260	420	500

a Find the best fit cubic equation that models this data. (3)

b Comment on the suitability of the best fit equation from part (a) as a model for the given data. (2)

c Using this model, find the week in which the book seller can expect to sell the most books, and state the number of books they can expect to sell in that week. (2)

d Find the week in which the book seller can expect to sell the least quantity books, and state the number of books they can expect to sell in that week. (2)

e Find the number of weeks per year that the book seller can expect to sell over 250 books. (3)

15 P2: After the strategy board game 'Chess' was invented, it is thought that the King of India (who loved the game) invited the inventor to ask for any reward for his great invention. The inventor asked that he be given a number of grains of rice, according to the following rule: 1 grain of rice on the first square of the chess board, 2 grains of rice on the second square of the chess board, 4 grains of rice on the third square, 8 grains of rice on the fourth square, 16 grains on the fifth, and so on until the last square (which is the 65th). If 1 kg of rice is estimated to contain around 20 000 grains, work out the number of tonnes of rice that the inventor obtained. Give your answer in standard form. (7)

16 P2: Consider the curve with equation $f(x) = 3x^3 - 9x^2 - 21x + 5$ and the points with coordinates $A(3, a), B(b, 5)$, where $a, b \in \mathbb{R}, b > 0$, which both lie on the graph of the above curve.

a Determine whether or not the origin lies on the curve. (1)

b Find the values of a and b. (5)

c Work out the length of the distance AB to the nearest unit. (2)

17 P2: A quadratic function $y = ax^2 + bx + c$ intersects the points with coordinates $(0, -18)$, $(-1, -10)$ and $(3, -30)$

a Find the values of a, b and c. (5)

b Solve the equation $ax^2 + bx + c = 0$. (3)

c Hence write down the equation of the line of symmetry of the graph of $y = ax^2 + bx + c$. (2)

18 P1: **a** Prove that $\log(1 - \frac{1}{2}) + \log(1 - \frac{1}{3}) + \log(1 - \frac{1}{4}) + \ldots + \log(1 - \frac{1}{n}) = -\log(n)$, for every $n \in \mathbb{N}$. (4)

b Prove that $\log_a(b^2) \times \log_b(a^3) = 6$, for every $a, b > 0$. (4)

c Consider an arithmetic sequence with first term $\ln(2)$ and second term $\ln(8)$. Prove that the sum of the first n terms of this sequence is given by $S_n = n^2 \times \ln(2)$. (4)

19 P1: A TV show gives participants the chance of winning some prizes. Each participant selects four boxes from a display of 10 boxes. The boxes are chosen completely at random.

Six of the boxes contain prizes, but four are empty.

a Find the probability that participant Peter wins 4 prizes. (2)

b Find the probability that participant Mary selects 4 empty boxes. (2)

c Given that Theresa has already selected 2 empty boxes, find the probability that she will then select two boxes containing prizes. (2)

20 P3: *In this question, you will determine parameters that allow a mathematical model of a real-life situation to best match experimental data.*

Ali is conducting an experiment to find an equation linking a particle's displacement (x metres), under the action of several forces, with the time (t seconds) for which it has been moving.

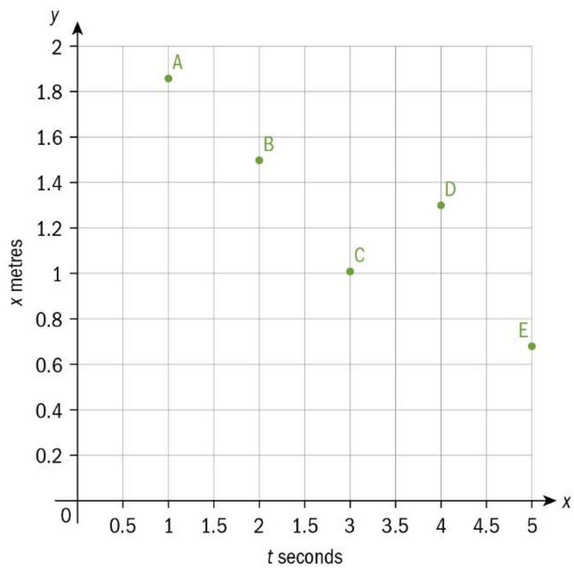
Ali suggests that the equation linking x and t is of the form

$$x = \frac{a}{bt+1}, \quad a, b > 0$$

In this equation, a and b are parameters which are dependent on the particular forces acting on the particle.

Ali measures the displacement of the particle every second from t=1 to t=5. He records his results in a table and then draws a scatter graph, as shown below.

Point	A	B	C	D	E
t (sec)	1	2	3	4	5
x (m)	1.86	1.50	1.01	1.3	0.68



a Explain why Ali might choose not to include point D in his analysis. (1)

Ali chooses to take just two points on his scatter graph to try to determine values of the parameters a and b.

b Use Ali's equation linking x and t, along with the points A and B, to form two linear equations and solve them to find values for a and b. (5)

c For this model, find the sum of the squares of the residuals at the points A, B, C and E.

(3)

Ali then uses a different method to find the parameters of the equation. He first rewrites the equation $x = \frac{a}{bt+1}$ in the form $f(x) = mt + c$, and then uses least squares linear regression to find the values of m and c.

d **i** Rearrange Ali's model into the form $f(x) = mt + c$ and state the function f that you obtain.

ii Find the least squares regression line for $f(x)$ on t, omitting the point D.

iii Hence find the values of a and b for this model. (10)

e **i** Find the sum of the of the squares of the residuals of the points A, B, C and E for the model found in part **d.**

ii Based on the analysis of squares of residuals, state whether the model found in part **b** or the model found in part **d** is the better model, justifying your answer.

iii Give a reason why the second method will usually give the better result. (5)

Answers

1 a $1.8 \le w < 1.9$ A1

b Using mid-points M1

GDC $\Rightarrow 1.85$ kg M1A1

c GDC $\Rightarrow 0.136$ kg M1A1

d Total frequency $= 201$ A1

101^{st} value lies in $1.8 \le w < 1.9$ M1A1

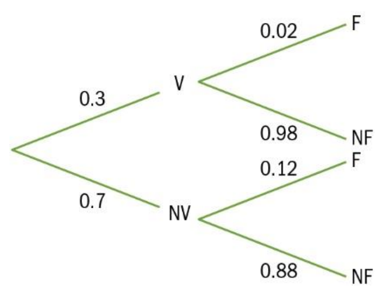
e Mean and median lie in same group $(1.8 \le w < 1.9)$ R1

6 standard deviations = $6 \times 0.136 = 0.816$ A1

and range $= 2.1 - 1.5 = 0.6$ A1

0.816 and 0.6 are close $\Rightarrow$ possible that the weights are normally distributed R1

2 a



(A3)

b $0.3 \times 0.02 = 0.006$ M1A1

c $P(NV \mid F) = \dfrac{P(NV \cap F)}{P(F)}$ M1

$= \dfrac{0.7 \times 0.12}{(0.3 \times 0.02) + (0.7 \times 0.12)}$ A1

$= \dfrac{0.084}{0.09}$

$= 0.933$ A1

3 a If they were mutually exclusive, then $P(A \cap B) = 0$, A1

but since they are independent, we have $P(A \cap B) = P(A)P(B) \neq 0$.

Therefore we have a contradiction, and so A and B are not mutually exclusive. R1

b i $P(A \cap B) = P(A)P(B) = 0.3 \times 0.8 = 0.24$ M1A1

ii $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.8 - 0.24 = 0.86$ M1A1

iii $P(A \mid B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{P(A) - P(A \cap B)}{P(B')} = \dfrac{0.3 - 0.24}{0.2} = \dfrac{0.06}{0.2} = 0.3$ M1A1

iv $P(A' \cap B) = P(B) - P(A \cap B) = 0.8 - 0.24 = 0.56$ M1A1

4 **a** $t = 0 \Rightarrow f(0) = 2.1$ m A1

b $f(t) = 0 \Rightarrow 3.10$ s M1A1

c Attempt to solve $2.1 + 14.5t - 4.9t^2 = 12.1$ M1

$t = 1.865,\ t = 1.094$ A1

$1.865 - 1.094$ M1

$= 0.770$ seconds A1

5 $1850 - 85 = 1765$ m A1

$\tan 6.5^\circ = \dfrac{1765}{x}$ M1

$x = \dfrac{1765}{\tan 6.5^\circ}$ A1

$x = 15\,491$ m M1

$x = 15.5$ km A1

6 **a** Applying the cosine rule to ΔBCD M1

$BD^2 = 12.3^2 + 10.9^2 - 2 \times 12.3 \times 10.9 \times \cos 120^\circ$ A1

$BD = \sqrt{404.17} = 20.10$ A1

Applying the cosine rule to ΔDAB M1

$\cos \angle DAB = \dfrac{13.1^2 + 17.9^2 - 404.17}{2 \times 13.1 \times 17.9}$ A1

$\Rightarrow \angle DAB = 79.2^\circ$ A1

b Applying sine rule for area to ΔDAB and ΔBCD M1

$\text{Area} = \left(\dfrac{1}{2} \times 13.1 \times 17.9 \times \sin 79.2^\circ\right) + \left(\dfrac{1}{2} \times 12.3 \times 10.9 \times \sin 120^\circ\right)$ A1A1

$= 173$ cm^2 A1

7 **a** $(3k + 1) - 19 = 19 - (5k - 3)$ M1A1

$3k - 18 = 22 - 5k$

$40 = 8k$

$k = 5$ A1

b $d = (3k + 1) - 19$ (or $d = 19 - (5k - 3)$) M1

$= 16 - 19$

$= -3$ M1

c $a = 22$ A1

Use of $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ M1

$-1980 = \frac{n}{2}\left[44 - 3(n-1)\right]$ A1

Use of GDC in Table mode (M1)

$n = 45$ A1

8 **a** $y = 5$ A1

b Gradient $BC = \frac{6-2}{10-1} = \frac{4}{9}$ M1

Gradient of perpendicular bisector $= -\frac{9}{4}$ A1

Mid-point $BC = \left(\frac{1+10}{2}, \frac{2+6}{2}\right) = \left(\frac{11}{2}, 4\right)$ A1

Equation bisector $y - 4 = -\frac{9}{4}\left(x - \frac{11}{2}\right)$ M1

$y = \frac{131}{8} - \frac{9}{4}x$

$8y + 18x - 131 = 0$ A1

c Setting $y = 5$ in the equation from **b** M1

$40 + 18x - 131 = 0$

$x = \frac{91}{18}$ A1

So $M\left(\frac{91}{18}, 5\right)$ A1

d Gradient $AC = \frac{6-8}{10-1} = -\frac{2}{9}$ M1

Gradient bisector $= \frac{9}{2}$ A1

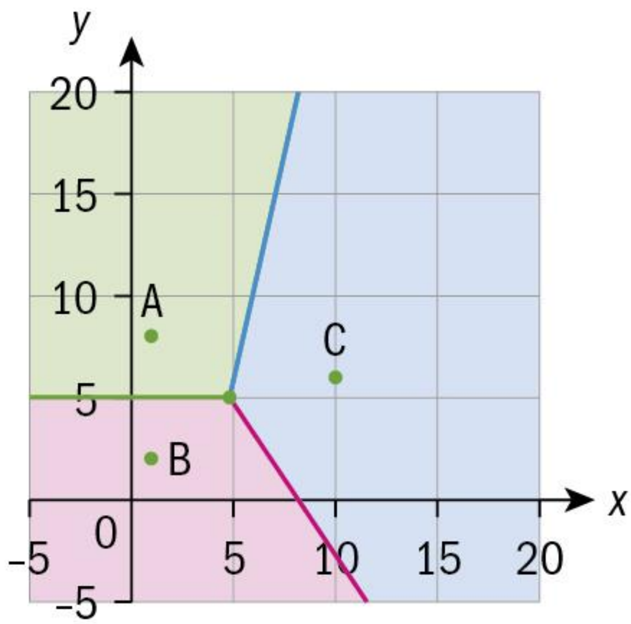
Mid-point $AC = \left(\frac{1+10}{2}, \frac{8+6}{2}\right) = \left(\frac{11}{2}, 7\right)$

Equation bisector $y - 7 = \frac{9}{2}\left(x - \frac{11}{2}\right)$ M1

$y - 5 = \frac{9x}{2} - \frac{99}{4}$

$18x - 4y - 71 = 0$ A1

e



A3

9 a $g \propto \frac{1}{x^2}$

$g = \frac{k}{x^2}$ M1

$k = gx^2 = 24.79 \times \left(7.1492 \times 10^7\right)^2$ A1

So $g = \frac{24.79 \times \left(7.1492 \times 10^7\right)^2}{x^2}$ A1

Substituting $x = 7.1492 \times 10^7 + 100\,000 = 7.1592 \times 10^7$ M1

$g = \frac{24.79 \times \left(7.1492 \times 10^7\right)^2}{\left(7.1592 \times 10^7\right)^2}$ A1

$= 24.7 \text{ m s}^{-2}$ A1

b Require field strength of 0.2479 m s^{-2}

$x = \sqrt{\frac{k}{g}}$ (1)

$x = \sqrt{\frac{24.79 \times \left(7.1492 \times 10^7\right)^2}{0.2479}}$

$x = 7.1492 \times 10^8$ (1)

So distance above surface is $7.1492 \times 10^8 - 7.1492 \times 10^7$ (1)

$= 6.43 \times 10^8 \text{ m}$

$= 6.43 \times 10^5 \text{ km}$ (1)

10 a $b = a + d$ A1

$c = a + 2d$ A1

b $\frac{a+c}{2} = \frac{a+(a+2d)}{2}$ M1

$= \frac{2a+2d}{2}$ A1

$= a + d = b$ A1A1

11 a his is an A.P. with $a_1 = 1, d = 2$ M1

$S_{200} = \frac{200}{2} \times (2 \times 1 + (200 - 1) \times 2) = 40\ 000$ A1

b $a_1 = 17,\ a_n = 379$ A1

$n = 182$ A1

$S_{182} = \frac{182}{2} \times (2 \times 17 + (182 - 1) \times 2)$ M1A1

$= 36\ 036$ A1

12 a arithmetic sequence with: $a_1 = 1, d = 1, n = 200$ A1

$S_1 = \frac{200}{2}(2 \times 1 + (200 - 1)1) = 20\ 100$ M1A1

b S_2 : arithmetic sequence with: $a_1 = 4, d = 4, a_n = 200$ A1

Finds: $n = 50$ A1

$S_2 = \frac{50}{2}(2 \times 4 + (50 - 1)4) = 5100$ A1

S_3 : Arithmetic sequence with: $a_1 = 9, d = 9, a_n = 198$ A1

Finds: $n = 22$ A1

$S_3 = \frac{22}{2}(2 \times 9 + (22 - 1)9) = 2277$ A1

S_4 :Arithmetic sequence with: $a_1 = 36, d = 36, a_n = 180$ A1

Finds: $n = 5$ A1

$S_4 = \frac{5}{2}(2 \times 36 + (5 - 1)36) = 540$ A1

c answer = $S_1 - S_2 - S_3 + S_4$ M1

We subtract multiples of 4, and subtract multiples of 9, but in this way we subtract twice the multiples of 36 (as 36 = 4 x 9). So, we add the multiples of 36 to the result. R1

answer = 13 263 A1

13 a $5100 \times (1 + 0.02)^3$ M1

answer = £5412 A1

b $100\,000 = 10\,000 \times (1 + 0.025)^n$ M1

By applying the logarithmic function to both sides, obtain: 93 years. A1

14 a $y = 440 - 43.2x + 2.24x^2 - 0.0269x^3$ M1A1A1

b The cubic model from part **a** appears largely suitable to model this data since most points are very close to the curve obtained R1

and the coefficient of determination is $R^2 = 0.985$ which indicates a very strong correlation. R1

c Week 43. 572 books sold A1A1

d Week 12. 196 books sold A1A1

e Weeks $0-6$ and $20-52$ M1M1

So 38 weeks A1

15 Geometric Sequence with: $a_1 = 1, r = 2, n = 64$ A1

$S_{64} = 1 \times \dfrac{2^{64} - 1}{2 - 1} = 2^{64} - 1 = 1.844674407 \times 10^{19}$ grains of rice M1A1

Divide by 20 000 to convert into kg: $\dfrac{1.844674407 \times 10^{19}}{20{,}000} = 9.22337237 \times 10^{14}$ kg M1A1

Divide by 1000 to convert to tonnes: $\dfrac{9.22337237 \times 10^{14}}{1{,}000} = 9.22337237 \times 10^{11}$ ton M1A1

16 a $3 \times 0^3 - 9 \times 0^2 - 21 \times 0 + 5 = 5 \neq 0$, hence no. A1

b $a = 3 \times 3^3 - 9 \times 3^2 - 21 \times 3 + 5 = -58$ A1

$5 = 3 \times b^3 - 9 \times b^2 - 21 \times b + 5$ M1

Obtain: $3b^3 - 9b^2 - 21b = 0$

$3b^2 - 9b - 21 = 0$, as $b > 0$ A1

$b = \dfrac{3 + \sqrt{37}}{2}$, $b = \dfrac{3 - \sqrt{37}}{2}$ (rejected due to $b > 0$) A1R1

$b = \dfrac{3 + \sqrt{37}}{2}$

c $AB = \sqrt{\left(\dfrac{3 + \sqrt{37}}{2} - 3\right)^2 + (-58 - 5)^2}$

$AB \approx 63.0189$

AB =63 units (to the nearest unit). A1

17 a $c = -18$ A1

$-10 = a - b - 18$

$-30 = 9a + 3b - 18$ A1

Attempt to solve simultaneously A1

$a=1$, $b=-7$ A1A1

b $(x+2)(x-9)=0$ or use GDC M1

$x=-2$ and $x=9$ A1A1

c $\frac{-2+9}{2}=\frac{7}{2}$ M1

$x=\frac{7}{2}$ A1

18 a LHS = $\log(\frac{1}{2})+\log(\frac{2}{3})+\log(\frac{3}{4})+\ldots+\log(\frac{n-1}{n})$ M1

$=\log(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\ldots\times\frac{n-1}{n})$ by Law of Logarithms M1

Notice that every denominator is being cancelled with every next numerator. R1

$=\log(\frac{1}{n})=\log(n^{-1})=-\log(n)$ A1

b $\log_a(b^2)=\frac{\log(b^2)}{\log(a)}$ M1

$\log_b(a^3)=\frac{\log(a^3)}{\log(b)}$ M1

LHS = $2\times\frac{\log(b)}{\log(a)}\times 3\times\frac{\log(a)}{\log(b)}=6$ M1A1

c Find the common difference $d=\ln(8)-\ln(2)=\ln(\frac{8}{2})=\ln(4)=2\ln(2)$ M1A1

Substitute into formula for the sum: $S_n=\frac{n}{2}(2\ln(2)+(n-1)2\ln(2))$ M1

Simplify to prove: $S_n=\frac{n}{2}(2n\ln(2))=n^2\ln(2)$ A1

19 a $\frac{{}^{6}C_4}{{}^{10}C_4}=0.0714(3sf)$ M1A1

b $\frac{1}{{}^{10}C_4}=0.00476(3sf)$ M1A1

c $\frac{{}^{6}C_2}{{}^{8}C_2}=0.536(3sf)$ M1A1

20 a Data which follows a function of the form $x=\frac{a}{bt+1}$ would mean x-values continually decrease as t increases. However, point D is greater than point C, so Ali might assume it is an outlier and exclude it. R1

b $1.86=\frac{a}{b\times 1+1}\Rightarrow a-1.86b=1.86$ M1A1

$1.5 = \dfrac{a}{2b+1} \Rightarrow a - 3.0b = 1.5$ A1

$a = 2.45,\ b = 0.316$ (M1)A1

c $x = \dfrac{2.45}{0.316t+1}$

Evidence of correctly identifying the residuals, e.g. $x(3) = \dfrac{2.45}{0.316 \times 3 + 1} = 1.26$,

$x(5) = \dfrac{2.45}{0.316 \times 5 + 1} = 0.95$ M1

Sum of square of the residuals =

$0 + 0 + (1.26 - 1.01)^2 + (0.95 - 0.68)^2 = 0.1354$ M1A1

d **i** $x = \dfrac{a}{bt+1}$ take reciprocal of both sides to give $\dfrac{1}{x} = \dfrac{b}{a}t + \dfrac{1}{a}$. M1A1

$f(x) = \dfrac{1}{x}$ A1

ii

Point	A	B	C	E
t (sec)	1	2	3	5
$\dfrac{1}{x}$ (m)	0.538	0.667	0.990	1.47

(M1)(A1)

$\dfrac{1}{x} = 0.242t + 0.251$ M1A1

iii $\dfrac{1}{a} = 0.251 \quad \Rightarrow a = 3.98$ M1A1

$\dfrac{b}{a} = 0.242 \Rightarrow b = 0.963$ A1

e **i** $x = \dfrac{3.98}{0.963t+1}$

Sum of square of the residuals =

$(2.02 - 1.86)^2 + (1.36 - 1.50)^2 + (1.02 - 1.01)^2 + (0.68 - 0.68)^2 = 0.045$

M1A1

ii The sums of squares of residuals for the model in **d** is smaller than the sums of squares of residuals for the model in **b.** R1

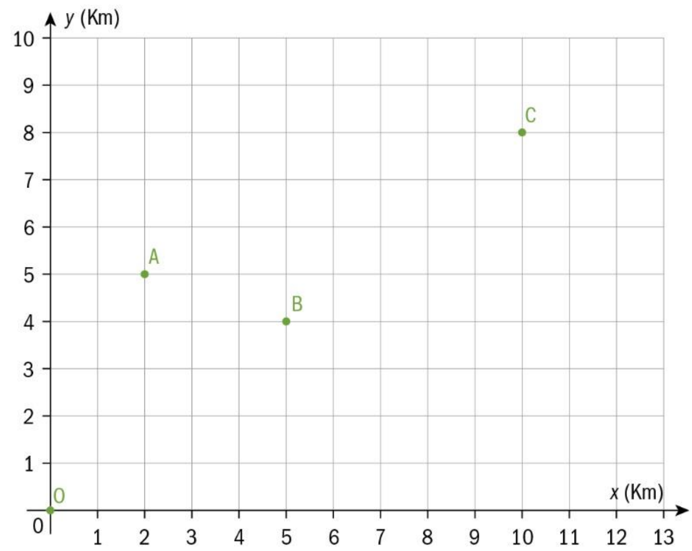
Hence the is the better model in **d** is the better one. A1

iii It uses all the data points to evaluate the parameters (or it uses a least squares technique to minimize the sum of the squares of the residuals), whereas the first method only used two data points to evaluate the parameters. R1

Exam practice: chapters 1-11

1 P1: Consider the diagram below, that shows the relative position of three villages A, B and C. The scale along the axes represents distances in kilometres.

The origin represents the location of a TV antenna.



a State the coordinates of A, B and C. (3)

b Find the distance between the villages A and C. (2)

The signal of the TV antenna cannot reach beyond 10 km.

c Show that this signal reaches village A, but not village C. (2)

d Show that B is the midpoint of the segment OC. (2)

e Hence deduce whether the signal reaches village B. (1)

2 P2: The records below show the number of days on which students were absent from a school during the first semester.

Number of absences	0	1	2	3	4	5	6
Number of students	78	23	12	7	4	3	1

a State the number of students in the school. (1)

b Find the percentage of students that were absent for 2 or more days. (2)

c Find the median number of absences. (2)

d Calculate the mean and standard deviation of the number of absences. (3)

e Draw a box-and-whisker plot to show the data. (3)

3 P1: The heights of 50 plants of the same species was recorded. The data was organised, and presented in the table below.

Height (cm)	$10 \le h < 14$	$14 \le h < 18$	$18 \le h < 22$	$22 \le h < 26$	$26 \le h < 30$
Frequency	7	14	12	n	8

a State the value of n. (1)

b Find an estimate for

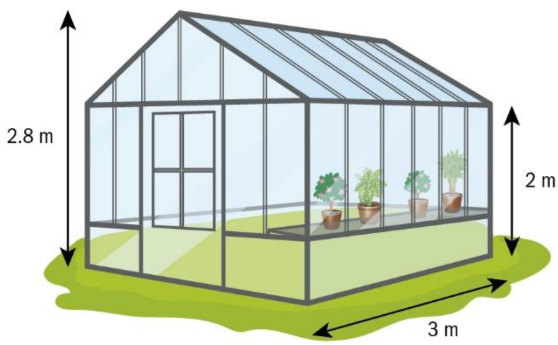
i the mean height of these plants

ii the standard deviation of the heights of the plants. (4)

c Draw a cumulative frequency curve that shows the distribution of the heights of the plants. (3)

d Draw appropriate dashed lines on your cumulative frequency curve from part **c** and hence estimate the median height of the plants. (2)

4 P2: Diagram not to scale



Mrs. Smith´s greenhouse has a maximum height of 2.8 metres.

The two vertical, rectangular sides are 2 metres high and 3 metres long.

The roof consists of two equal rectangles, each of which has an area of 6 m^2.

a Find the dimensions of the rectangles that make the roof. (2)

b Hence find the width of the greenhouse. (3)

c Determine the total surface area of all the vertical sides of the greenhouse. (3)

d Calculate the volume of the greenhouse. (3)

5 P2: The amount of caffeine, $C\,ml$, contained in the bloodstream of an adult has a half-life of 4.9 hours. Jose initially consumed 200 ml of caffeine.

a Determine the exact values of the parameters A and k in the expression $C(t) = A\left(2^{-kt}\right)$ that models the amount of caffeine in Jose´s blood, t hours after consuming the caffeine. (4)

b Determine how long it will take until the amount of caffeine in Jose´s blood is less than 20 ml. (2)

c Find the rate of change in the amount of caffeine in Jose's blood, at time $t = 10$ hours.

(3)

d State, with a reason, the value that C tends towards as t tends to infinity. (2)

6 P2: The rent-a-car company *More is Less* offers discounts to businesses. The size of the discount is proportional to the number of cars rented by the business each year.

The basic price to rent a business car for one year is 2500 euros. The percentage discount for each car is 1% of the rental price, multiplied by the total number of cars rented by that business, for up to 60 cars.

a Show that the total cost to a business renting x cars is given by

$C(x) = 2500x - 25x^2$, $0 \le x \le 60$. (2)

b Find the total cost to a business renting

i 20 cars **ii** 40 cars. (2)

c Find an expression for $\frac{dC}{dx}$. (2)

d Hence, find the maximum value of $C(x)$ and the value of x for which this occurs. (3)

The company *Xport* always rents between 40 and 60 cars per year.

e Explain how the result you obtained in part **d** may cause *Xport* to decide to rent 60 cars per year. (2)

7 P2: Market surveys' data indicate that the model for the supply function of a certain product is of the form $s(x) = Ax^2 + Bx + C$ where s represents the amount (or supply) of product available, measured in thousands of units; A, B, C are parameters to be determined; and x is the price of the product, in euros.

Suppliers were asked which quantities they were willing to supply at different prices. The results are shown in the table below:

Price offered (euros)	*Quantity supplied (Thousands of units)*
25	1125
30	2500
40	6000

a Write down three equations relating the parameters A, B and C . (3)

b Verify that $A = 5, B = 0$ and $C = -2000$ satisfy the 3 equations you wrote down in part **a**. (1)

c Show that if the price was 20 euros, this model would predict that suppliers would not be willing to produce this product. (2)

The same survey asked consumers about how much they were willing to pay for the same product. Researchers created the model for the demand function as $d(x) = x^2 - 100x + 2500$, where d represents the quantity demanded, measured in thousands of units, if the price of the product was x euros.

d Solve the equation $s(x) = d(x)$, giving your answer correct to two decimal places. (3)

e Hence find how many units of this product can be completely sold for the price found in **d**. (3)

8 **P1:** Data collected in a city indicates that the average daily number of crimes is related to the number of police officers on duty each day. It is modelled by the function $C(x) = 400 - 9.5x\left(2^{-0.02x}\right)$ where x represents the number of police officers on duty and C represents the average number of crimes.

a Determine the average number of crimes expected if

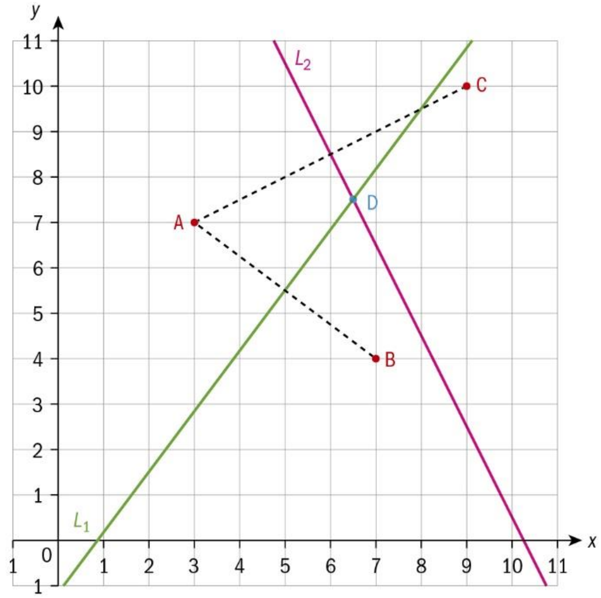
i there are no police officers on duty

ii there are 40 police officers on duty. (3)

b Find the number of police officers on duty, according to the model, if the average number of crimes is a minimum. State also what the minimum average number of crimes is. Give your answers to the nearest integer. (3)

c State, with a reason, what the model implies happens to the average number of crimes as $x \to \infty$. (2)

9 **P2:** The Voronoi diagram shows the position of three schools A, B and C in a rural region.



a State the coordinates of the points A, B and C. (3)

Line L_1 and line L_2 are the perpendicular bisectors of the segments AB and AC respectively.

b Show that L_1 is defined by the equation $8x - 6y - 7 = 0$. (2)

c Find the equation of L_2. (3)

Let D be the point of intersection of L_1 and L_2.

d Hence find the coordinates of D. (2)

e Find the coordinates of M, the midpoint of BC. (1)

f Hence copy and complete the Voronoi diagram above. (4)

g If a water deposit was to be built at an equal distance to all three schools, state its location. (1)

h Shade the cells of the diagram that represents the points that are closer to C than to A and B. (1)

10 P2: Consider the function defined by $f(x) = x^3 - 9x$ for $-3 \le x \le 3$.

a Sketch the graph of f, showing clearly the axes intercepts. (2)

b Use the trapezoid rule, with strip width of 0.5, to find an approximation for $\int_0^3 f(x)\,dx$. (4)

c Find the exact value of $\int_0^3 f(x)\,dx$. (2)

d Explain why the approximation found in **a** is larger than the exact value found in **b**. (1)

e Write down, with a reason, the value of $\int_{-3}^3 f(x)\,dx$. (2)

11 P1: Calculate the area $A(m)$ of the region bounded by the graph of the function $f(x) = \frac{e}{x}$, the x-axis and the lines $x = 5$ and $x = m, m > 5$. Give your answer in the form $e\ln(k \times m)$, where k is a rational number to be found. (5)

12 P2: **a** Let $f(x)$ be a function. Write down a condition for $f(x)$ to be concave up. (1)

b By using differentiation, prove that the function $f(x) = \ln(x), x > 0$ is concave up on its domain. (5)

13 P1: Solve the following differential equations, giving your answers in the form $y = f(x)$ for some function of x.

a $y' = 4xy^2, y > 0$ (5)

b $y'y = x, y > 0$ (4)

c $y' = e^{-y}\cos(x)$ (4)

14 P1: If $f(x) = 4\cos(x)(\sin(x) + \cos(x))$, find the exact minimum and the maximum values of the function $f(x)$. (8)

15 P2: Given the complex number $z = \frac{1 + i\sqrt{3}}{2}$, prove that $z^{2000} = \frac{-1 + i\sqrt{3}}{2}$. (7)

16 P1: A rectangle has fixed perimeter P. Prove that the maximum area enclosed by this rectangle occurs when the rectangle is a square. (13)

17 P1: **a** Find the derivative of the function $f(x) = \ln(x + \sqrt{x^2 + 1})$. Simplify your answer as much as possible. (4)

b Hence, show that $\int_0^1 \frac{1}{\sqrt{x^2 + 1}}dx = \ln(1 + \sqrt{2})$. (4)

18 P1: Find the following integrals.

a $I_1 = \int \frac{x^2}{\sqrt{x^3 + 2}}dx$ (7)

b $I_2 = \int \frac{1}{x\sqrt{\ln(x)}} dx$ (7)

19 P2: **a** We consider a rectangle with length x and width y. Given that the length is increasing with a rate of 4 ms^{-1} and the width is increasing with a rate of 5 ms^{-1}, find the rate of change of the area of the rectangle at the instant when the length is 30 cm and the width is 40 cm. Give your answer in $\text{m}^2\text{ s}^{-1}$. (5)

b A snow ball is melting. The radius of the snow ball is given by $r = 5 - t^2$ metres, where t is time in seconds. Calculate the rate of change of the volume of the ball at $t = 2\text{ s}$.

(5)

c Consider points $A(x-1, 0)$ and $B(0, \ln(x))$, $x > 0$. Given that the rate of change of x is 3 cms^{-1}, calculate the rate of change of the distance AB when $x = 2\text{ cm}$. Give your answer correct to 3 significant figures. (5)

20 P3: *In this question, you will investigate the probability of a player winning in a game of chance if they stop playing when they reach a particular value, or if they lose all their money.*

This type of scenario has applications in many fields, including survival of small populations in an ecosystem.

Bethan is playing a game of chance. In each play of the game, the probability she wins is $\frac{1}{3}$ and the probability she loses is $\frac{2}{3}$. If she wins, she receives double the money she paid to play the game.

She begins with £2 and chooses to pay £1 for each play of the game. She resolves to play the game until she has £4, or loses all her money.

a Draw a tree diagram to show the possible outcomes for the first three plays of the game, indicating the amount of money Bethan would have at the end of each play. (3)

b Hence, find the probability Bethan will have

i reached £4 in the first three plays.

ii have £1 after the first three plays. (4)

c Explain why Bethan must play an even number of games. (2)

The probability of Bethan reaching £4 can be written in the form $a + ar + ar^2 + \ldots$

($a, r \in \mathbb{R}$).

d **i** Find the values of *a* and *r*.

ii Hence find the probability Bethan reaches £4. (6)

An alternative strategy Bethan considers is to place £2 on a single play of the game. As before, she resolves to play the game until she has £4, or loses all her money.

e Write down the probability she reaches £4. (1)

f Give one reason why she might choose to play each of the two strategies. (2)

Bethan is given £1 by a friend, so she now has £3. She decides she will pay £1 for each play of the game but finds that the method used in part **d** to calculate the probability of winning £4 no longer applies. Instead, she decides to use a matrix approach.

Let the money Bethan is currently holding represent the initial state in a transition matrix.

g Write down the 5×5 transition matrix (T) for a single play of the game. (3)

h Explain why the money Bethan possesses at each stage forms a Markov chain. (1)

A Markov chain which has a state that once entered, cannot be left, is said to have an absorbing state.

i State the two absorbing states for the Markov chain. (1)

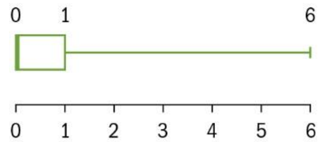
It is given that for a Markov chain with absorbing states the matrix T^n will give the probabilities of moving from one state to another in n steps.

j Find the probability Bethan reaches £4 if she has £3 initially. (3)

k Find the probability she reaches £4 if she has £3 initially and pays £1 on the first play and if she loses, £2 on the second. (2)

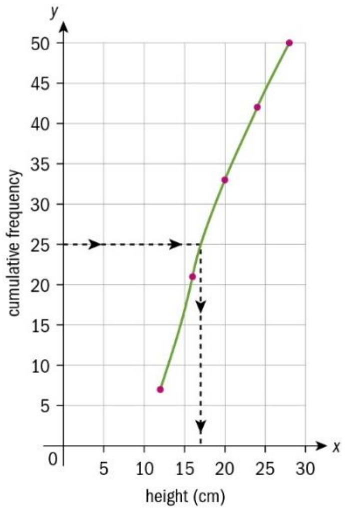
Answers

1	**a**	A(2, 5), B(5,4) and C(10,8)	A1A1A1
	b	$\sqrt{8^2+3^2}=8.54\,km(3sf)$	M1A1
	c	$\text{OA}=\sqrt{2^2+5^2}=5.38\ldots<10,\qquad \text{OC}=\sqrt{10^2+8^2}=12.8\ldots>10$	R1R1
		So signal can reach A but not C	AG
	d	$\frac{0+10}{2}=5;\frac{0+8}{2}=4$	A1A1
		so B is mid-point of OC	AG
	e	$\text{OB}=\frac{1}{2}\text{OC}=6.4\ldots<10$ so signal reaches B	R1
2	**a**	128	A1
	b	$\frac{27}{128}\times100\%=21.1\%(3sf)$	M1A1
	c	0	A2
	d	$\mu=0.820(3sf),\sigma=1.32(3sf)$	M1A1A1
	e	Min=Q1=Med=0, Q3=1, Max=6	M1A1



A1

3	**a**	$n=50-(7+14+12+8)=9$	A1
	b	Taking the midpoints of the intervals	
		i 19.76 **ii** 5.15 (3sf)	M1A1A2
	c		



d Using the cumulative frequency graph (as shown with dashed lines) M1 estimate for median is 17.1 cm A1

4 a $6 \div 3 = 2$

2 metres by 3 metres A1A1

b $2\sqrt{2^2 - 0.8^2} = 3.67$ M1A1A1

c $2\left((3\times 2)+\left(\dfrac{3.66..\times 0.8}{2}\right)+3.66..\times 2\right)$ M1A1

$= 29.6\ \text{m}^2$ A1

d $3\times\dfrac{3.66..\times 0.8}{2}+2\times 3.66..\times 3$ M1A1

$= 26.4\ \text{m}^3$ A1

5 a $A = 200$ A1

$\dfrac{A}{2} = A\times 2^{-4.9k} \quad \Rightarrow 4.9k = 1 \Rightarrow \quad k = \dfrac{10}{49}$ M1A1A1

b Solving $200\times 2^{\frac{-10}{49}t} = 20$ $t = 16.3$ hours $(3sf)$ M1A1

c Using GDC, $\dfrac{dC}{dt}$ at $t = 10$ is $-6.88\,\text{ml}$ per hour $(3sf)$ M1A2

d $C = \dfrac{200}{2^{\frac{10t}{49}}}$ as $t \to \infty$ $2^{\frac{10t}{49}} \to \infty$ so $C \to 0$ R1A1

6 a $2500x - \textit{discount} = 2500x - \dfrac{1}{100}\times 2500x\times x = 2500x - 25x^2$ R1M1AG

b i $2500\times 20 - 25\times 20^2 = 40\,000$ euros

ii $2500\times 40 - 25\times 40^2 = 60\,000$ euros A1A1

c $\dfrac{\text{d}C}{\text{d}x} = 2500 - 50x$ M1A1

d $\dfrac{\text{d}C}{\text{d}x} = 2500 - 50x = 0 \Rightarrow \text{max at } x = 50,\ C = 62\,500$ euros M1A1A1

e Due to the symmetry of the quadratic curve about $x = 50$, the cost of renting (for example) 60 cars is the same as the cost of renting 40 cars. R1

So if they need any more than 40 cars, it would be cheaper to rent 60. R1A1

7 a $625A + 25B + C = 1125$, $900A + 30B + C = 2500$, $1600A + 40B + C = 6000$ A1A1A1

b $625\times 5 - 2000 = 1125$, $900\times 5 - 2000 = 2500$, $1600\times 5 - 2000 = 6000$ M1

c $5\times 20^2 - 2000 = 0$ so supply is 0 at this price M1A1

d $5x^2 - 2000 = x^2 - 100x + 2500 \Rightarrow 4x^2 + 100x - 4500 = 0 \Rightarrow x = 23.29$ euros M1A1A1

e Need to find the supply available when the cost per unit is 23.294… euros. R1

$5\times 23.294...^2 - 2000 = 713(3sf)$ thousand units M1A1

8 a i 400 **ii** $400 - 9.5 \times 40 \times 2^{-0.02\times 40} \approx 182$ A1 M1A1

b $x \approx 72$ $C \approx 148$ M1 A1A1

c $C \to 400$ A1

because $2^{-0.02x} \to 0$ faster than $x \to \infty$, so the second term tends to zero. R1

9 a A(3,7), B(7,4), C(9,10) A1A1A1

b All points (x, y) on L_1 are equidistant from A and B, so

$\sqrt{(x-3)^2 + (y-7)^2} = \sqrt{(x-7)^2 + (y-4)^2}$ M1

$\Rightarrow x^2 - 6x + 9 + y^2 - 14y + 49 = x^2 - 14x + 49 + y^2 - 8y + 16$ A1

$\Rightarrow 8x - 6y - 7 = 0$ AG

c $\sqrt{(x-3)^2 + (y-7)^2} = \sqrt{(x-9)^2 + (y-10)^2}$ M1

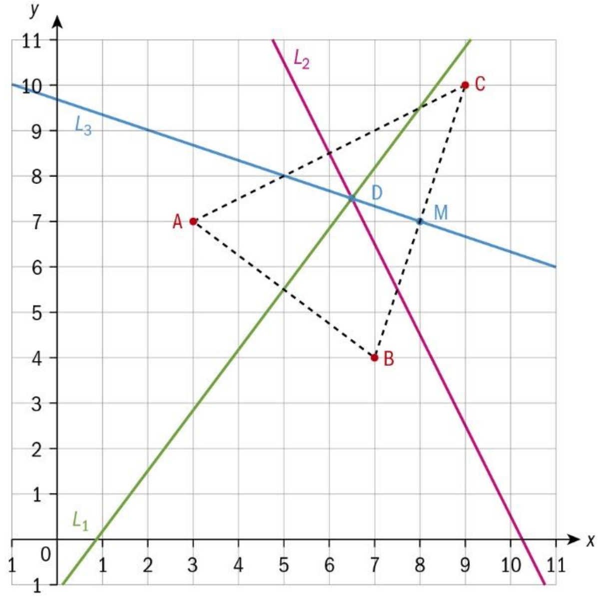
$\Rightarrow x^2 - 6x + 9 + y^2 - 14y + 49 = x^2 - 18x + 81 + y^2 - 20y + 100$ A1

$\Rightarrow 12x + 6y - 123 = 0 \Rightarrow 4x + 2y - 41 = 0$ A1

d Solving the 2 equations gives D(6.5,7.5) M1A1

e M(8,7) A1

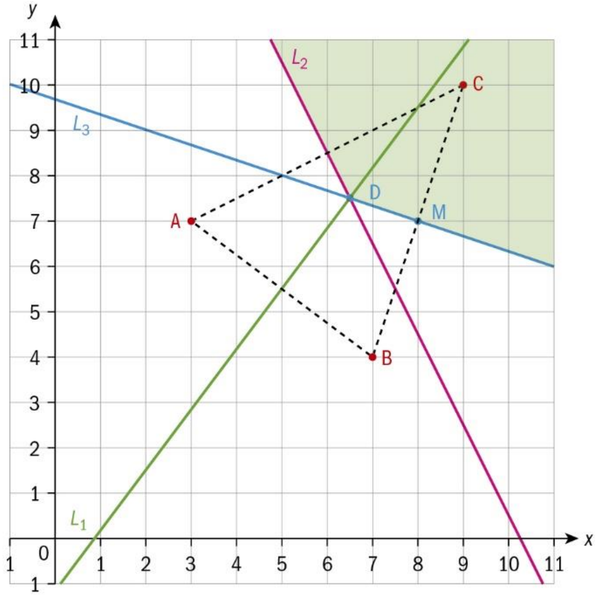
f 3rd perpendicular bisector will go through D and M R1



A3

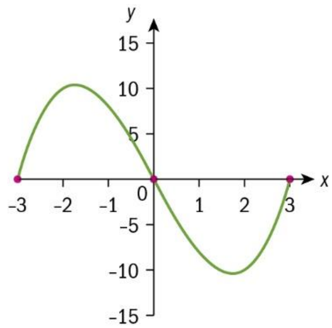
g D A1

h



A1

10 a



A1A1

b $0.5(-4.375-8-10.125-10-6.875)=-19.7(3sf)$ M1A1A2

c $\int_0^3 x^3-9x\,dx=-20.25$ M1A1

d As the graph is concave up, the trapeziums below the x-axis lie entirely within the region bounded by the curve, so will miss some of the area. R1

e 0 A1

the graph has rotational symmetry about the origin such that the area above the x-axis between $-3\le x\le 0$ is the same size as the area below the x-axis between $0\le x\le 3$ and the two cancel out. R1

11 $A(m)=\int_5^m \frac{e}{x}dx$ M1

$A(m)=e\left[\ln(x)\right]_5^m$ A1

$A(m)=e(\ln(m)-\ln(5))$ M1

$A(m) = e\ln(\frac{m}{5})$, $k = \frac{1}{5}$ A2

12 a $f(x)$ is concave up when $f''(x) < 0$ A1

b Finds first derivative: $f'(x) = \frac{1}{x}$ A1

Finds second derivative: $f''(x) = \frac{-1}{x^2}$ M1A1

Deduces that the second derivative is negative for all $x > 0$. R1

So, by part **a**, $f(x)$ is a concave function. A1

13 a Separable variables: $y^{-2}dy = 4xdx$ M1A1

integrate: $\int y^{-2}dy = \int 4xdx$ M1

$\frac{y^{-1}}{-1} = 2x^2 + c, c \in \mathbb{R}$ A1

$y = \frac{-1}{2x^2 + c}$ A1

b $\frac{dy}{dx} = \frac{x}{y}$ A1

Separable variables & integrate: $\int ydy = \int xdx$ M1

$\frac{y^2}{2} = \frac{x^2}{2} + c, c \in \mathbb{R}$ A1

$y = \pm\sqrt{x^2 + 2c}$ A1

c $\frac{dy}{dx} = e^{-y}\cos(x)$ A1

Separable variables & integrate: $\int e^y dy = \int \cos(x)dx$ M1

$e^y = \sin(x) + c, c \in \mathbb{R}$ A1

$y = \ln(\sin(x) + c)$ A1

14 Expand the brackets: $f(x) = 2\sin(2x) + 4\cos^2(x)$ M1

Use $\cos^2(x) = \frac{\cos(2x)+1}{2}$ to write $f(x) = 2\sin(2x) + 2\cos(2x) + 2$ M1A1

Write $2\sin(2x) + 2\cos(2x) + 2 = R\sin(2x + a)$ (or equivalently $2\sin(2x) + 2\cos(2x) + 2 = R\cos(2x + a)$) M1

$R\sin(2x + a) = R\sin(2x)\cos(a) + R\cos(2x)\sin(a)$, hence deduce that $\begin{matrix} R\cos(a) = 2 \\ R\sin(a) = 2 \end{matrix}$ M1A1

By squaring the latter two equations, obtain: $R^2 = 8 \Rightarrow R = 2\sqrt{2}$ A1

Hence, $f_{\min}(x) = 2 - 2\sqrt{2}$
$f_{\max}(x) = 2 + 2\sqrt{2}$ A1

15 Write in Modulus-argument form: $z = 1 \times (\cos(60^\circ) + i\sin(60^\circ))$ A1

Use of De Moivre's Theorem: $z^{2000} = 1^{2000} \times (\cos(60^\circ \times 2000) + i\sin(60^\circ \times 2000))$ M1A1

Obtain: $z^{2000} = \cos(120\,000^\circ) + i\sin(120\,000^\circ)$

$= \cos(360^\circ \times 333^\circ + 120^\circ) + i\sin(360^\circ \times 333^\circ + 120^\circ)$ M1

$= \cos(120^\circ) + i\sin(120^\circ)$ M1A1

Hence, $z^{2000} = \dfrac{-1 + i\sqrt{3}}{2}$ A1

16 Denotes as x, y the length and width of the rectangle respectively.

Expresses perimeter $P = 2x + 2y$ M1A1

Make y (or equivalently x) the subject: $y = \dfrac{P - 2x}{2}$ M1

Expresses area $A = x \times y$ M1

Obtains $A = (\dfrac{P - 2x}{2})x = \dfrac{Px - 2x^2}{2}$ A1

Finds the derivative: $\dfrac{dA}{dx} = \dfrac{P}{2} - 2x$ M1A1

Sets equal to zero and solves to find x which at turning point of A :

$\dfrac{P}{2} - 2x = 0$ M1

$x = \dfrac{P}{4}$ A1

Finds second derivative: $\dfrac{d^2A}{dy^2} = -2 < 0$, so this turning point is a maximum R1

Finds y when $x = \dfrac{P}{4}$: $y = \dfrac{P - 2 \times \dfrac{P}{4}}{2} = \dfrac{P}{4}$ M1A1

So, max area when $x = y = \dfrac{P}{4}$, which is a square R1

17 a $f'(x) = \dfrac{1}{x + \sqrt{x^2 + 1}} \times (1 + \dfrac{1}{2\sqrt{x^2 + 1}} \times 2x)$ M1A1

$f'(x) = \dfrac{1}{x + \sqrt{x^2 + 1}} \times (\dfrac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}})$ M1

$f'(x) = \dfrac{1}{\sqrt{x^2 + 1}}$ A1

b $\int_0^1 \frac{1}{\sqrt{x^2+1}}dx = \int_0^1 f'(x)dx$ M1

$\int_0^1 \frac{1}{\sqrt{x^2+1}}dx = f(1) - f(0)$ M1

$= \ln(1+\sqrt{1^2+1}) - \ln(0+\sqrt{0^2+1})$ A1

$= \ln(1+\sqrt{2})$ A1

18 a Uses substitution: $u = x^3 + 2$

M1 for attempting a substitution, A1 for correct substitution

$\Rightarrow \frac{du}{dx} = 3x^2$ M1 for differentiating

$\Rightarrow dx = \frac{du}{3x^2}$ A1 for correctly making dx the subject

$I_1 = \int \frac{x^2}{\sqrt{u}} \times \frac{1}{3x^2} du$ M1

$I_1 = \frac{1}{3}\int \frac{1}{\sqrt{u}} du = \frac{2}{3}\sqrt{u}$ A1

$I_1 = \frac{2}{3}\sqrt{x^3+2} + c, c \in \mathbb{R}$ A1

b Uses substitution: $u = \ln(x)$

M1 for attempting a substitution, A1 for correct substitution

$\frac{du}{dx} = \frac{1}{x}$ M1 for differentiating

$dx = xdu$ A1 for correctly making dx the subject

$I_2 = \int \frac{1}{x\sqrt{u}} \times xdu = 2\int \frac{1}{2\sqrt{u}} du$ M1

$I_2 = 2\sqrt{u} + c, c \in \mathbb{R}$ A1

$I_2 = 2\sqrt{\ln(x)} + c, c \in \mathbb{R}$ A1

19 a Area A : $A = x \times y$ M1

Applies product rule: $\frac{dA}{dt} = \frac{dx}{dt} \times y + x \times \frac{dy}{dt}$ M1A1

Converts centimetres into metres: $x = 0.3$ m, $y = 0.4$ m M1

Substitutes: $\frac{dA}{dt} = 4 \times 0.4 + 0.3 \times 5 =$ 3.1 m^2 s^{-1} M1A1

b Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Differentiates: $\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt}$ M1A1

$\frac{dr}{dt} = -2t = -2 \times 2 = -4\ \text{m s}^{-1}$ A1

$r = 5 - 2^2 = 1\ \text{m}$ M1

Substitutes: $\frac{dV}{dt} = \frac{4}{3}\pi \times 3 \times 1^2 \times (-4) = -16\pi\ \text{m}^3\ \text{s}^{-1}$ M1A1

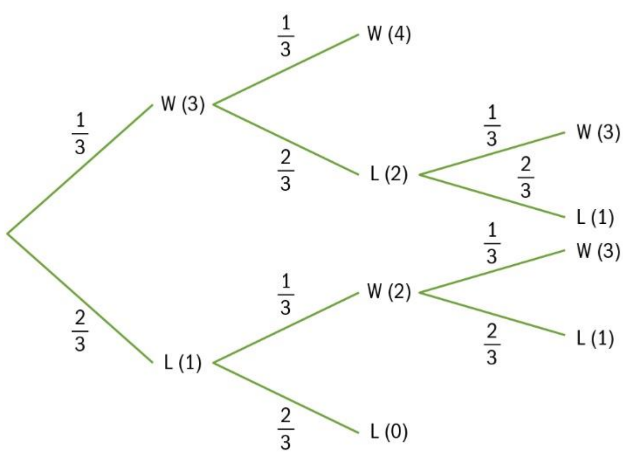
c Expresses the distance of the points:
$d(x) = \sqrt{(x-1-0)^2 + (0-\ln(x))^2} = \sqrt{(x-1)^2 + \ln^2(x)}$ M1A1

Differentiates: $\frac{d(d(x))}{dt} = \frac{1}{2\sqrt{(x-1)^2 + \ln^2(x)}} \times (2(x-1) \times \frac{dx}{dt} + 2\ln(x) \times \frac{1}{x} \times \frac{dx}{dt})$ M1A1

Substitutes: $\frac{d(d(x))}{dt} = \frac{1}{2\sqrt{(2-1)^2 + \ln^2(2)}} \times (2(2-1) \times 3 + 2\ln(2) \times \frac{1}{2} \times 3)$ M1A1

Answer: $\frac{d(d(x))}{dt} = 3.32\ \text{cm s}^{-1}$ A1

20 a



M1A1A1

b i $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ M1A1

ii $2 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$ M1A1

c It is not possible to win after just one play given Bethan has £2 initially. R1

After two plays of the game Bethan will have either finished playing or will again have £2. R1

This cycle repeats itself for further plays of the game. Hence it will take an even number of plays for Bethan to either reach £4 or lose all her money. AG

d i Let X be the number of plays for Bethan to reach £4.

P(Bethan reaches £4) $= \text{P}(X = 2) + \text{P}(X = 4) + \text{P}(X = 6) + \ldots$ M1

$= \frac{1}{9} + \left(2 \times \frac{2}{3} \times \frac{1}{3}\right) \times \frac{1}{9} + \left(2 \times \frac{2}{3} \times \frac{1}{3}\right)^2 \times \frac{1}{9} + \ldots$ M1A1

$a = \frac{1}{9}, \; r = \frac{4}{9}$ A1

ii $S_\infty = \frac{a}{1-r} = \frac{\frac{1}{9}}{1 - \frac{4}{9}} = \frac{1}{5}$ M1A1

e Probability she reaches £4 is $\frac{1}{3}$. A1

f She may choose the second strategy as she has a higher chance of winning £4. R1
But with the second strategy, she only gets to play the game once. Hence she may choose the first. R1

g $\begin{pmatrix} 1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1 \end{pmatrix}$ M1A1A1

h The probability of being in a state (having a fixed amount of money) is dependent only her previous state. R1

i £4 and £0 A1

j For high values of n matrix approximates to $\begin{pmatrix} 1 & \frac{14}{15} & \frac{4}{5} & \frac{8}{15} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{15} & \frac{1}{5} & \frac{7}{15} & 1 \end{pmatrix}$ (M1)(A1)

Probability she reaches £4 $= \frac{7}{15}$ A1

k $\frac{1}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{5}{9}$ M1A1

Exam practice: chapters 1-15

1 P1: The following table shows the probability distribution for a random variable X .

x	1	2	3	4	q
$P(X=x)$	0.1	0.3	0.15	0.2	r

a Find the value of r . (2)

b Find the value of q, given $E(X)=4.75$. (2)

c Find $P(X>3.5)$ (2)

d Find $P(X>2 \mid X \leq 4)$ (2)

2 P1: A driving test consists of two sections: a practical test and a theory test.

The following table shows the results of ten candidates taking the test, each marked as a percentage.

Practical	66	70	52	26	59	66	61	81	48	60
Theory	85	70	35	33	56	66	62	91	69	82

a Write down a table of ranks for this data. (3)

b Calculate Spearman's rank correlation for this data. (2)

c Briefly comment on your result. (2)

3 P1: Seven athletes from Kingsley Harriers ran a 400 m race. Their times were compared with similar athletes from Hammonds' Hounds. The athletes' times (in s) are shown in the following table.

In this question, you will use a suitable hypothesis test to determine if, on average, the athletes from Hammonds' Hounds can run a 400 m race quicker than athletes from Kingsley Harriers.

Kingsley Harriers	66.1	55.6	58.3	59.9	57.3	57.2	54.2
Hammonds' Hounds	65.9	53.1	56.8	56.7	56.0	54.4	54.0

a Write down the null and alternative hypotheses. (2)

b Perform a t-test at the 5% significance level. (3)

c Write down the conclusion to the test. (2)

d Comment on what the conclusion to the t-test would be if performed at the 10% significance level. (1)

4 P2: The number of daily newspapers sold from a particular outlet is analysed over a typical week.

The information is recorded in the table below.

Day	Mon	Tues	Wed	Thurs	Fri
No. of newspapers sold	76	60	64	88	91

Perform a goodness of fit test at the 5% significance level to determine if the number of newspapers sold fits a uniform distribution.

You should state your null and alternative hypotheses and justify any conclusions found. The critical value for the test is 9.49. (7)

5 P1: The scores in a national Maths exam are approximately normally distributed with mean 53 and variance 121. The number of candidates taking the exam was 3560.

Find:

a the expected number of candidates that scored at least 80 marks, giving your answer to the nearest whole number. (2)

b the probability that a candidate selected at random would have scored 30 marks or less. (2)

c The minimum number marks a candidate would need to have achieved to be among the top 10%. (2)

6 P1: According to national records, the population of Vienna in 1880 was 726 000. By 1890, Vienna's population had grown to 1 365 000, and in 1910 it reached 2 031 000.

a Find the average rate of change of the population of Vienna between

i 1880 and 1890

ii 1890 and 1910. (2)

b Hence, justify that Vienna's population increase between 1880 and 1910 does not fit a linear model. (1)

The following table shows the changing population of Vienna in the 21st century.

Year	Population (in millions)
2000	1.55
2005	1.63
2010	1.69
2015	1.80
2016	1.84
2017	1.88

Let y represent the population in millions and t represent the number of years after 2000.

c Find the equation of the line of regression y on t that models the population of Vienna between the years 2000 and 2017. (3)

d Use this line of regression to estimate the population of Vienna in 2008. (2)

e By referring to the Pearson product-moment correlation coefficient r between y and t, comment on the appropriateness of the model found in part **c**. (2)

7 P1: *Coolwings* airline historical records suggest that the probability that a passenger does not turn up for a booked flight on a Saturday morning is 2%.

This Saturday morning, 153 passengers have booked to board a *Coolwings* flight. Let the discrete random variable X be the number of passengers who do not turn up for the flight.

a State the distribution that X satisfies, including the values of the parameters. (3)

b Determine the expected number of passengers who do not turn up for the flight. (1)

c Find the probability that at least three passengers do not show up. (2)

The plane's maximum capacity is 150 passengers. In case of overbooking, *Coolwings* has to pay $300 to each passenger that cannot board the flight.

d Determine the expected amount *Coolwings* has to pay due to overbooking, giving your answer to 2 decimal places. (3)

8 P1: A TV show gives participants the chance of winning some prizes. Each participant selects four boxes from a display of 10 boxes. The boxes are chosen completely at random.

Six of the boxes contain prizes, but four are empty.

a Find the probability that participant Peter wins 4 prizes. (2)

b Find the probability that participant Mary selects 4 empty boxes. (2)

c Given that Theresa has already selected 2 empty boxes, find the probability that she will then select two boxes containing prizes. (2)

9 P2: The blood pressure of 250 workers at the same company was measured. The results of the systolic blood pressure (bp), measured in mm of mercury, are shown in the table below.

Pressure (bp)	$bp \le 110$	$110 < bp \le 120$	$120 < bp \le 130$	$130 < bp \le 140$	$140 < bp \le 150$	$150 < bp$
Number of workers	16	50	74	68	26	16

The analyst who collected the data believes that the distribution of the blood pressure of the workers of this company follows a normal distribution with mean of 130 and standard deviation of 13. In this question, you will test the analyst's claim.

a State

i which test is appropriate to test this claim

ii the null hypothesis

iii the alternative hypothesis. (3)

b i Under H_0, calculate the expected frequencies. Give your answers to 2 decimal places, in similar table to that above.

ii State the number of degrees of freedom. (6)

c Carry out the test and state the $p-$value. (3)

d State, with a reason, the conclusion of the test at the 5% confidence level. (2)

10 P2: When subjected to radiation, a certain type of cell has exactly 3 possible reactions: cell dies, with probability of 0.5; cell splits into two cells, with probability 0.3; cell survives as a single cell.

a Write down the probability that a cell of this type survives as a single cell. (1)

Two cells of this type are independently subject to radiation. Let the discrete random variable X be the total number of living cells that exist after the experiment.

b By carefully justifying your argument, show that $\mathrm{P}(X=2)=0.34$. (3)

c Copy and complete the following table.

x	0	1	2	3	4
$P(X=x)$	0.25		0.34		

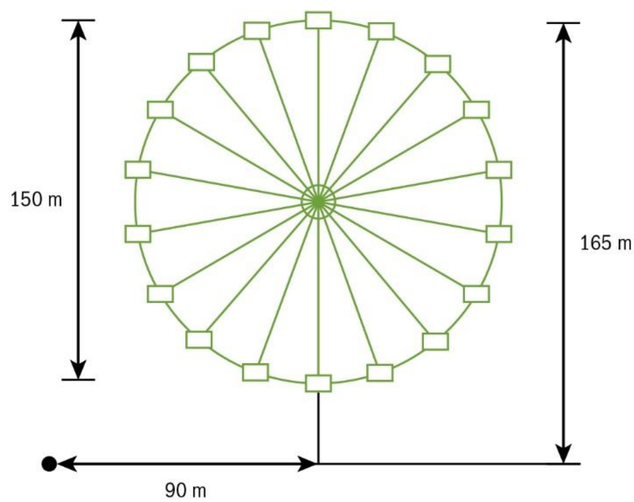
(6)

d Find

i $E(X)$

ii $\mathrm{Var}(X)$ (3)

11 P2: A Ferris Wheel of diameter 150 metres has height of 165 metres. The wheel has 28 cabins equally spaced along its circumference. Each cabin is travelling at a constant speed of 25 cm s^{-1} anticlockwise around the circumference.



a Find the length of the arc of the wheel´s circumference between two consecutive cabins. (2)

The cabins are numbered 1-28 in an anticlockwise order.

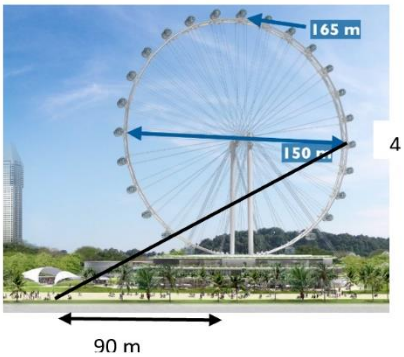
b Find the straight-line distance between cabins 1 and 4. (5)

At 12:00 cabin 4 is at a height of 90 metres and moving up.

c Determine the height of cabin 1 at 12:00. (4)

d Find how long it will take until cabin 1 reaches the height of 90 metres for the first time after 12:00. Give your answer correct to the nearest second. (5)

At 12:00 Kristin is lying on the ground 90 metres away from the base of the wheel on the opposite side to cabin 4, as shown in the diagram. Kristin and all the cabins lie in the same plane. She looks up in the direction of cabin 4.



e Find the angle of elevation from Kristin's position on the ground to the position of cabin 4 at that time. (3)

12 P2: A survey of customers at a bank suggests that, between 12:00 pm and 12:15 pm, an average of 6 people will enter a bank. To test this hypothesis, the manager goes outside of the bank between 12:00 pm and 12:15 pm and counts the number of people who enter the bank. The manager finds that 8 people enter the bank between 12:00 pm and 12:15 pm.

By using a level of significance of 5%, carry out an appropriate test to check whether the average number of people entering the bank is greater than what the survey predicts. (6)

13 P2: I have a six-faced die which I suspect that is a biased toward the number "5". To test out my suspicion, I perform a test: I throw the die 20 times and note that 8 of them were "5". By using an appropriate statistical test with significance level of 5%, deduce whether the die is biased or not. (6)

14 P2: In a factory, the probability of a new product passing the quality control tests is 90%. After some improvements that the management carried out on the construction process, they want to test out if there is any significant improvement in the percentage of products passing the quality control tests.

They picked 50 new products at random, and found that 49 did not pass quality control. Carry out an appropriate statistical test, with level of significance of 5%, to check whether there is improvement or not. (6)

15 P2: A new car costs £12 000 and its value V decreases after t years at a rate of £$\frac{4{,}000}{(t+1)^2}$ per year. Find the value of the car after 3 years. (7)

16 P2: Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$. (8)

17 P2: Consider the differentiable function $f(x)$ such that $f(x^2)+f(x)=3\ln(x)+2018$ for all $x \in \left]0,\infty\right[$.

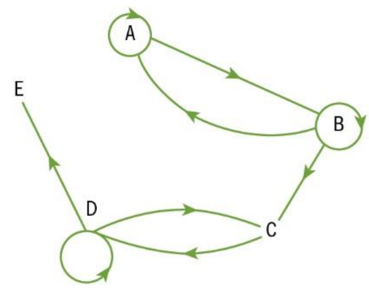
a Prove that $f(1)=1009$. (3)

b Find the value of $f'(1)$. (4)

c Find the equation of the tangent to $f(x)$ at the point where $x = 1$. Give your answer in the form $y = mx + c$, where $m, c \in \mathbb{R}$. (2)

18 P2: Two positive numbers add up to give 2018. Find the maximum value of their product. (10)

19 P2: **a** Find the adjacency matrix M for the following graph:



(2)

b Find the number of walks of length 2 from B to B. (4)

20 P3: The purpose of the question is to find a model which plots the outline of a nautilus shell on the Argand plane.

a Write the following numbers in the form $x + yi$ where $x, y \in \mathbb{R}$ and should be given as exact values.

i $2e^0$ **ii** $3e^{\frac{\pi}{4}i}$ **iii** $4e^{\frac{\pi}{2}i}$

iv $5e^{\frac{3\pi}{4}i}$ **v** $6e^{\pi i}$ (3)

There exist two real numbers, *a* and *b*, such that all the complex numbers given in part **a** can be written in the form $z = (a + b\theta)e^{\theta i}$.

b Find the values of *a* and *b.* (4)

c Show the five complex numbers from part **a** on an Argand diagram, along with the numbers $z = (a + b\theta)e^{\theta i}$ for which $\theta = \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \frac{7\pi}{2}, 4\pi, \frac{9\pi}{2}$. Join these points to make a spiral. (3)

Let $z_1, z_2, z_3 \ldots$ be the series of points where the spiral cuts the positive real axis and $u_1, u_2, u_3 \ldots$ be the series of points where the spiral cuts the positive imaginary axis.

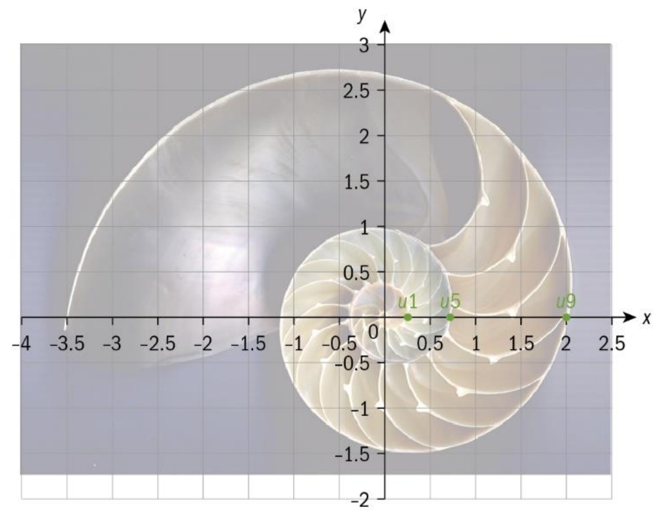
d **i** Explain why $z_1, z_2, z_3 \ldots$ forms an arithmetic sequence with a common difference of 8.

ii State the first term and common difference for the sequence $u_1, u_2, u_3 \ldots$

iii A straight line is drawn out from the origin. State the difference between the moduli of successive points where this line cuts the spiral. Explain what this means geometrically. (7)

Now you will find an expression to model the locus of the nautilus shell shown.

Three adjacent points are plotted where the spiral of the shell cuts the positive real axis. Let these points be u_1, u_5 and u_9 where $u_1 = 0.26$, $u_5 = 0.72$ and $u_9 = 2.0$.



e Explain why a spiral of the form $z = (a + b\theta)e^{\theta i}$ is not a good model for this nautilus shell. (1)

A model of the form $z = (ce^{\theta d})e^{\theta i}$, $c, d \in \mathbb{R}$ is suggested.

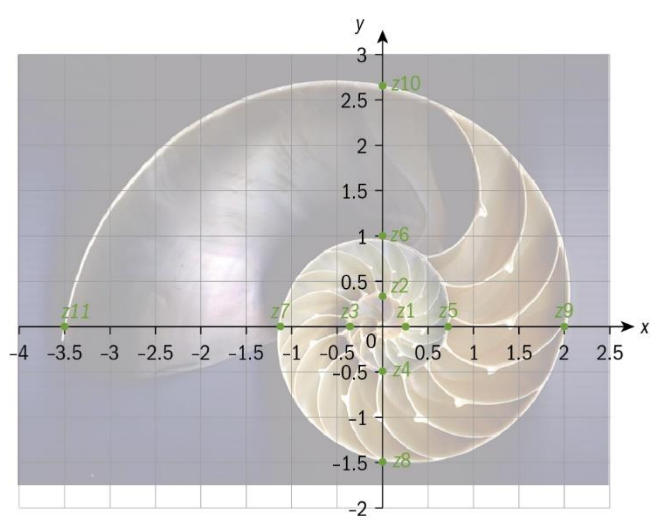
Let u_1 be the point where $\theta = 0$ and u_5 be the point where $\theta = 2\pi$.

f Use the points z_1 and z_5 to find approximate values for c and d, giving your answers to 2 significant figures. (3)

g **i** Use you answer to part **f** to find an estimate for the point z_9, where $\theta = 4\pi$.

ii The true value of z_9, as shown in the image, is $z_9 = 2.0$. Find the percentage error in your estimate from part **i**. (2)

In order to improve the model further points are taken where the spiral crosses the real and imaginary axes.



The modulus and the value of θ for each of the points is given in the table below.

Point (z)	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	z_{11}
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π	$\frac{9\pi}{2}$	5π
modulus $(\lvert z\rvert)$	0.26	0.33	0.36	0.5	0.70	1.0	1.13	1.5	2.0	2.66	3.5

h Find the least squares regression line of $\ln(\lvert z\rvert)$ against θ in the form $\ln(\lvert z\rvert) = f\theta + g,\ f, g \in \mathbb{R}$

i Hence find new estimates for the values of c and d.

j Use these values of c and d to estimate the value of u_9, where $\theta = 3\pi$. (3)

Answers

1 **a** $0.1+0.3+0.15+0.2+r=1$ M1

$r=0.25$ A1

b $1\times0.1+2\times0.3+3\times0.15+4\times0.2+0.25q=4.75$ M1

$q=11.2$ A1

c $0.2+0.25$ M1

$=0.45$ A1

d $P(X>2 \mid X\le 4)=\dfrac{P(3\le X\le 4)}{P(X\le 4)}$ M1

$=\dfrac{0.35}{0.75}$

$=\dfrac{7}{15}=0.467$ A1

2 **a**

Practical rank	7.5	9	3	1	4	7.5	6	10	2	5
Theory rank	9	7	2	1	3	5	4	10	6	8

A3

b $r_S=0.736$ M1A1

c r_S is positive and reasonably close to 1, R1

indicating a fairly strong positive correlation between a person's results

in their theory and practical tests. R1

3 **a** $H_0: \mu_1=\mu_2$ A1

$H_1: \mu_1>\mu_2$ A1

b $t-$ value $=0.765$ M1A1

$p-$ value $=0.229$ A1

c $0.229>0.05$ so accept H_0. R1A1

i.e. there is no significant evidence to suggest that the athletes from Hammonds' Hounds can run a 400 m race quicker than athletes from Kingsley Harriers.

d The conclusion would remain the same, since $0.229>0.1$. R1

4 H_0: The data satisfies a uniform distribution A1

H_1: The data does not satisfy a uniform distribution A1

$\chi^2_{\text{calc}}=10.142$ M1A1

$p-$ value $=0.038$ A1

$\chi^2_{\text{calc}}=10.142>9.49$, therefore we reject H_0 R1A1 i.e. There is sufficient evidence to suggest that the number of newspapers sold is not a uniform distribution

5 a $\sigma = 11$

$3560 \times P(X \geq 80) = 25$ M1A1

b $P(X \leq 30) = 0.0183(3sf)$ M1A1

c $P(X \leq T) = 0.9 \Rightarrow T = 67.1(3sf)$ M1A1

6 a i 63 900 people per year **ii** 33 300 people per year A1A1

b The gradients of the two lines connecting the 3 points are very different, so model cannot be a linear one. R1

c $y = 0.0187t + 1.54$ $(3sf)$, M1A2

d $0.0187 \times 8 + 1.54 = 1.69(3sf)$ millions M1A1

e The linear model is appropriate as the Pearson product-moment correlation coefficient $r = 0.986$ which indicates very strong positive correlation between time and the population of the city during the 21st century. A1R1

7 a $B(153, 0.02)$ A1A1A1

b $np = 3.06$ A1

c $P(X \geq 3) = 1 - P(X \leq 2) = 0.592(3sf)$ M1A1

d $P(X = 0) \times 900 + P(X = 1) \times 600 + P(X = 2) \times 300 = \$192.11$ M1A1A1

8 a $\frac{^6C_4}{^{10}C_4} = 0.0714(3sf)$ M1A1

b $\frac{1}{^{10}C_4} = 0.00476(3sf)$ M1A1

c $\frac{^6C_2}{^8C_2} = 0.536(3sf)$ M1A1

9 a i A χ^2 goodness of fit test A1

ii The blood pressure satisfies the $N(130, 13^2)$ distribution A1

iii The blood pressure does not satisfy this distribution A1

b i

Pressure (bp)	$bp \leq 110$	$110 < bp \leq 120$	$120 < bp \leq 130$	$130 < bp \leq 140$	$140 < bp \leq 150$	$150 < bp$
Expected frequency	15.49	39.73	69.78	69.78	39.73	15.49

M1A4

ii $6 - 1 = 5$ A1

c $p-\text{value}=0.172$ M1A2

d $0.172>0.05$ so we accept the null hypothesis and believe the analyst's claim. R1A1

10 a 0.2 since probabilities must add to 1 A1

b $X=2$ could be die/split or split/die or survive/survive R1

$0.5\times0.3+0.3\times0.5+0.2\times0.2=0.34$ M1A1

c $X=1$ could be die/survive or survive/die

$0.5\times0.2+0.2\times0.5=0.2$

$X=3$ could be split/survive or survive/split

$0.3\times0.2+0.2\times0.3=0.12$

$X=4$ is split/split

$0.3\times0.3=0.09$ M1A1M1A1M1A1

x	0	1	2	3	4
$P(X=x)$	0.25	0.2	0.34	0.12	0.09

Check probabilities add to 1

d **i** 1.6 **ii** $1.23288...^2=1.52(3sf)$ A1M1A1

11 a $\frac{2\pi\times75}{28}=16.8299...=16.8m(3sf)$ M1A1

b angle at centre between cabins 1 and 4 is $360\times\frac{3}{28}$ A1

Let distance be x.

$x^2=75^2+75^2-2\times75\times75\cos\left(360\times\frac{3}{28}\right)\Rightarrow x=49.5418...=49.5m(3sf)$ M1A1A2

c The bottom of the wheel is 15 m above the ground, since 165-150=15.

Cabin 4 is 90 m above the ground, so is 90 – 15 = 75 m above the lowest point of the wheel. Since the radius of the wheel is also 75 m, the line from cabin 4 to the centre of the wheel is horizontal. R1

Height is $90-75\sin\left(360\times\frac{3}{28}\right)=43.2m(3sf)$ M1A1A1

d Arc distance to travel is 3×16.8299 speed is 0.25×60 m per minute A1A1

$\text{time}=\frac{16.8299...\times3}{0.25\times60}=3.366=3\text{mins}\,22\text{secs}$ M1A1A1

e Vertical distance is 90, horizontal distance is 90+75=165 A1

Angle is $\arctan\frac{90}{165}=28.6^\circ(3sf)$ M1A1

12 Let X be the random variable denoting the number of people entering the bank.

It follows Poisson Distribution with parameter $\lambda = 6$. R1

$H_0 : \lambda = 6$
$H_1 : \lambda > 6$ A1

$P(X \geq 9) = 1 - P(X < 8) = 1 - 0.84724 = 0.15276 > 0.05$ M1A1R1

So, accept null hypothesis. The average number of people entering the bank is 6. A1

13 Let X be the random variable that counts the number of "5"s.

$X \sim B(2, 20, \frac{1}{6})$. R1

$H_0 : p \leq \frac{1}{6}$, die NOT biased

$H_1 : p > \frac{1}{6}$, die biased A1

$P(x \geq 2) = 1 - P(x < 2) = 1 - B(8, 20, \frac{1}{6}) = 0.01125 < 0.05$ M1A1R1

So, reject the null hypothesis, and conclude the die is biased. A1

14 X : random variable that counts successful products. $X \sim B(49, 50, 0.9)$. A1

$H_0 : p \leq 0.9$, no improvement

$H_1 : p > 0.9$, improvement A1

$P(X > 0.9) = 0.03378 < 0.05$ M1A1R1

So, reject H_0, so there is improvement. A1

15 $\frac{dV}{dt} = -\frac{4000}{(t+1)^2}$ (must use a negative sign, since the value decreases) A1

$\int dV = \int -\frac{4000}{(t+1)^2} dt$ M1

$V(t) = \frac{4000}{t+1} + c$ A1

$V(0) = 12\,000 \Rightarrow 12{,}000 = \frac{4{,}000}{0+1} + c$ M1

$c = 8000$ A1

Finds: $P(3) = \frac{4000}{3+1} + 8000$ M1

$P(3) = £9000$ A1

16 Finds the eigenvalues of matrix $M = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix}$:

$\det\begin{pmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 36 = 0$ M1

Finds: $\lambda = -5$ or $\lambda = 7$ A1

Finds the corresponding eigenvector for $\lambda = -5$: $\begin{pmatrix} 1-(-5) & 12 \\ 3 & 1-(-5) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ M1

Gets: $v_1 = -2v_2$ A1

Finds: $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is eigenvector corresponding to $\lambda = -5$ A1

Finds the corresponding eigenvector for $\lambda = 7$: $\begin{pmatrix} 1-7 & 12 \\ 3 & 1-7 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ M1

Gets: $v_1 = 2v_2$ A1

Finds: $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is eigenvector corresponding to $\lambda = 7$ A1

17 a Substitutes $x = 1$: $f(1^2) + f(1) = 3\ln(1) + 2018$ M1

$2f(1) = 2018$ A1

$f(1) = 1009$ AG

b Differentiates using chain rule M1

$f'(x^2) \times 2x + f'(x) = \dfrac{3}{x}$ A1

Substitutes $x = 1$: $f'(1^2) \times 2 \times 1 + f'(1) = \dfrac{3}{1}$ M1

$3f'(1) = 3 \Rightarrow f'(1) = 1$ A1

c $y - f(1) = f'(1)(x-1)$

$y - 1009 = 1 \times (x-1)$ M1

$y = x + 1008$ A1

18 x, y the two numbers so: $x + y = 2018$ M1

Denotes as product of them: $P(x,y) = x \times y$ M1

$P(x) = x \times (2018 - x)$ M1

Differentiates: $\dfrac{dP}{dx} = 2018 - 2x$ M1

At x_{max} $\dfrac{dP}{dx} = 0$ M1

$2018 - 2x_{max} = 0$ M1

Finds: $x_{max} = 1009$ A1

Verifies that this is max: $\dfrac{d^2P}{dx^2} = -2 < 0$ R1

Finds: $y = 2018 - 1009 = 1009$ A1

So product is $1009^2 = 1\,018\,081$ A1

19 a

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ M1A1

b Attempts to multiply M^2 M1

Finds: $$M^2 = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ A1

Writes that the required is the entry at the 2nd row and the 2nd column. M1

Answers: 2 A1

20 a **i** 2 **ii** $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}\mathrm{i}$

iii 4i **iv** $-\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}\mathrm{i}$

v -6 M1A1A1

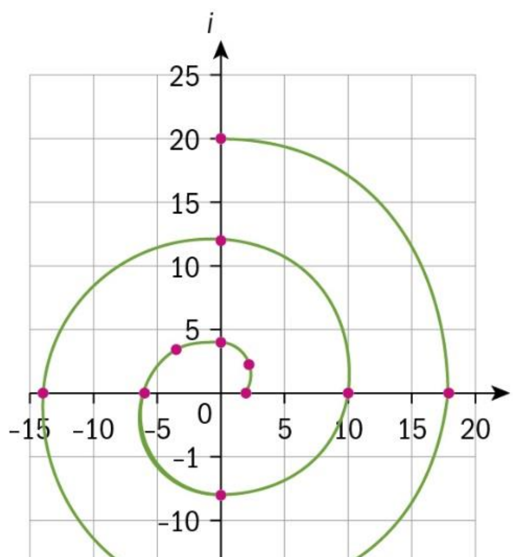
b From **i,** $2 = a + b \times 0 \Rightarrow a = 2$

From **ii,** $3 = 2 + b\frac{\pi}{4} \Rightarrow b = \frac{4}{\pi}$

Check in **iii:** $2 + \frac{4}{\pi} \times \frac{\pi}{2} = 4$ which is correct M1M1

So $a = 2,\ b = \frac{4}{\pi}$ A1A1

c



M1A1A1

d **i** For each successive term in the sequence, θ increases by 2π R1

and (since $e^{n2\pi i} = 1$ for all n) each term increases by $\frac{4}{\pi} \times 2\pi = 8$ R1

ii First term is 4i, common difference is 8i A1A1

iii Difference is 8, A1

this means that the **distance** A1

between successive points where the line cuts the curve will **always be 8**, A1

(independent of the angle at which the curve is drawn).

e The nautilus does not intersect a straight line at successive points which are a constant distance apart, which was the case for a spiral of the form $z = (a + b\theta)e^{\theta i}$. R1

f $u_1 = 0.26$ and $u_5 = 0.72$

Hence $0.26 = ce^0 = c$ A1

$0.72 = 0.26e^{2\pi d}$ M1

$\left(d = \frac{1}{2\pi}\ln\left(\frac{0.70}{0.26}\right)\right)$

$d = 0.158\ldots \approx 0.16$ A1

g **i** $u_9 \approx 0.26e^{4\pi \times 0.158} = 1.89$ A1

ii $\frac{0.11}{2.0} \times 100 = 5.5\%$ A1

h $\ln(|u|) = 0.169\theta - 1.42$ M1A1A1

i $\ln c = -1.42 \Rightarrow \quad |u| = 0.24e^{0.169\theta}$ M1A1

j 2.00 A1